REPORT OF THE WAR PREPAREDNESS COMMITTEE OF THE AMERICAN MATHEMATICAL SOCIETY AND MATHEMATICAL ASSOCIATION OF AMERICA AT THE HANOVER MEETING

In fulfilling its mission to prepare the Society and Association to serve the country in time of war the Committee recognizes three objectives

(1) The solution of mathematical problems essential for military or naval science, or rearmament.

(2) The preparation of mathematicians for research essential for objective (1).

(3) The strengthening of undergraduate mathematical education in our colleges to the point where it affords adequate preparation in mathematics for military and naval services of any nature. A study by a large group of mathematicians of the current routine military texts and sources wherever mathematics is involved in order that mathematicians may exert their proper influence on the teaching of military and naval science in time of war.

Organization. It is recommended that the present Subcommittee on Education be replaced by two new subcommittees corresponding to objectives (2) and (3) respectively and that the present Subcommittee on Research remain unchanged. So reorganized there would be three subcommittees designated as follows:

I. Committee on Research
II. Committee on Preparation for Research
III. Committee on Education for Service.

It is recognized that Committee III corresponds very closely to the special interests and aims of the Association. It is recommended that chairmen of Committees II and III be appointed at Hanover but that the completion of these committees be left until the new chairmen have had time to make recommendations as to the personnel of their committees.

Consultants. Having particular regard to the solution of problems, we recommend the appointment of Chief Consultants in each of the following fields: Ballistics, Aeronautics, Computation (numerical, mechanical, electrical), Cryptanalysis, Industry, Probability and Statistics. Other fields may well be added. The Chief Consultants would be responsible in research matters to the Research Committee without themselves taking on any of the executive duties of that committee. These consultants should also help the Committee on Preparation for Research when called upon. The three central committees should remain small so that they can act quickly. It is also recommended that there be *ordinary consultants* in each of the above fields. These ordinary consultants would be recommended for appointment so as to obtain some sort of correspondence with the subfields. Thus able young mathematicians so appointed might prepare themselves in advance to help in a given field. These consultants should also be appointed so as to meet the needs of particular sections of the country for help in its industries. It is suggested that any mathematician who feels competent in a field might ask to be made a consultant, and might be accepted as such if that seemed desirable. A flexible group is indicated.

Duties

1. *Committee on Research.* Problems would be received and assigned to consultants when available, otherwise assigned to members of the Research Committee. Recommendations as to educational needs would be made to the Committee on Education for Research. Suitable contacts would be made by this committee. The proper assignment and orientation of young consultants should be the duty of this committee.

2. *Education for Research.* This committee would continue work already done on bibliographies, texts, seminars, publications, etc. Professor A. W. Tucker, representing the Annals of Mathematics Studies, has stated that there will be no difficulty in publishing suitable texts concerning applications of mathematics in war or industry. New texts in practically all of the above fields are needed and competent mathematicians have already been invited to write such texts. A small loan fund may be necessary to pay the initial cost of such Studies since these texts are published on a cash basis without profit.

3. *Education for Routine Service.* The War Department intends in general to use sections of existing texts for instruction in the military schools. Those few texts which involve mathematics in an essential way should be carefully read by members of this committee, with two objects in view.

(1) *What elementary mathematics must be taught in order that these books be readily understood?*

(2) *In what way could the material in these books of a mathematical nature be better written to save time and achieve the desired results?*

The following three resolutions are recommended for adoption by the Council of the A.M.S. and Board of Governors of the M.A.A.:

1. That all competent students in the secondary schools take the maximum amount of mathematics available in their institutions. In the case of many schools, additions to the present curriculum will be necessary in order to furnish an adequate background for the military needs of the country.

2. That the colleges and universities at once make such revisions of their undergraduate courses in mathematics and add such courses to the curriculum as are necessary to prepare students in the elements of mechanics, probability, surveying, navigation and other essentials of military science.

3. That the graduate schools extend their courses in applied mathematics, such as dynamics, hydrodynamics, elasticity, aeronautics, ballistics, statistics, and that advanced students be urged to become highly qualified in one or more fields of applied mathematics.

These resolutions should be given immediate publicity in order that the changes recommended may be undertaken at once. In connection with resolutions 2 and 3 Committees II and III will be able to give a certain amount of advice at once and in a few months will undoubtedly be able to help considerably more.

We are appending two bibliographies corresponding respectively to the needs of those primarily interested in research and those primarily interested in education for service. These bibliographies are selective rather than exhaustive because it is felt that such bibliographies will best serve immediate needs. More extensive bibliographies are available and will be mailed to those interested upon request. The report of the War Preparedness Committee in extenso containing letters from various authorities and other important source material may be had upon writing to Secretary Richardson. Needless to say the Committee will help those who are working on war preparedness problems in any way in which it can.

Prepared for the Committee by

MARSTON MORSE
General Chairman

The above three resolutions were adopted by the Council of the American Mathematical Society and the Board of Governors of the Mathematical Association of America at a joint meeting at Hanover, N. H.

Professor M. H. Stone has been appointed Chairman of the Subcommittee on Preparation for Research and Professor W. L. Hart has been appointed Chairman of the Subcommittee on Education for Service. Chief Consultants have been appointed as follows:

Ballistics—Professor John von Neumann, Institute for Advanced Study
Aeronautics—Professor Harry Bateman, California Institute of Technology.
Computation—Professor Norbert Wiener, Massachusetts Institute of Technology.
Industry—Dr. T. C. Fry, Bell Telephone Laboratories
Probability and Statistics—Professor S. S. Wilks, Princeton University
Cryptanalysis—Professor H. T. Engstrom, Yale University.

Service Bibliography

Seacoast Artillery Gunnery, F. M. 4–10. U. S. Government Printing Office. Price 25 cents.

Seacoast Artillery, F. M. 4–15 and 4–10. U. S. Government Printing Office. To be published.

Gunnery, Fire Control and Position Finding for Anti-Aircraft Artillery. Special text No. 26, U. S. Government Printing Office. (*Restricted*)

Field Artillery, F. M. 6–40. 1939, U. S. Government Printing Office, Washington, D. C. Price 25 cents.

Field Artillery. Gunnery, 1936, Book 161. Printed: Field Artillery School, Fort Sill, Oklahoma.

Signal Communications, F. M. 24–5. U. S. Government Printing Office, Washington, D. C. Price 45 cents.

Practical Air Navigation. Special Publication No. 197, U.S. Department of Commerce, Coast and Geodetic Survey (temporarily out of print)

The Use of the Airway Radio Range and Other Radio Aids. Free publication of Civil Aeronautics Authority, Washington, D. C.

Aerodynamics. Bulletin No. 21, Civil Aeronautics Authority, Washington, D. C.

Anti-Aircraft Artillery. Gunner's Instruction, IX Expert Gunners, Coast Artillery Journal, 1115 17th Street, N.W., Washington, D. C.

Sound Ranging. F. M. 6–120, U. S. Government Printing Office, Washington, D. C.

Meteorology. Army Extension Courses, Special Text No. 193, U. S. Government Printing Office, Washington, D. C. Price 20 cents.

Coast Artillery Journal. $4 a year. 1115 17th St. N.W., Washington D. C.

Ordnance

T. J. Hayes. *Elements of Ordnance*, John Wiley and Sons, 1938. Chapters X, XI, XII, Exterior Ballistics, Probability of Hitting, Bombing from Airplanes, written by Kent, Dederick, Zornig, of Aberdeen Proving Ground.

Aerodynamics

B. Jones. *Elements of Practical Aerodynamics*, Wiley and Sons.
C. C. Carter. *Simple Aerodynamics*, Ronald Press.

Navigation

Dutton. *Navigation and Nautical Astronomy*, U. S. Naval Institute.
P. V. H. Weems. *Air-Navigation*, McGraw-Hill.

Optics

McKinley. *Applied Aerial Photography*, Wiley.
Reeves. *Aerial Photographs, Characteristics and Military Applications*, Ronald.
Hotine. *Surveying from Air Photographs*, Constable.
Gruber. *Photogrammetry*, Chapman.

Meteorology

McAdie. *Principles of Aerography*, Rand.
D. S. Piston. *Meteorology*, P. Blakiston and Son.

RESEARCH BIBLIOGRAPHY

Ballistics

T. J. Hayes. *Elements of Ordnance*, John Wiley and Sons. 1938. Chapters X, XI, XII, Exterior Ballistics, Probability of Hitting, Bombing from Airplanes, written by Kent, Dederick, Zornig, of Aberdeen Proving Ground.

F. R. Moulton. *New Methods in Exterior Ballistics*, University of Chicago Press, 1926.

R. H. Fowler and others. *The Aerodynamics of a Spinning Shell*, Philosophical Transactions of the Royal Society of London, series A, vol. 221 (1921), pp. 295–387, and vol. 222 (1922), pp. 227–247.

A. E. H. Love and F. B. Pidduck. *Lagrange's Ballistic Problem*, Phil. Trans., vol. 222 (1922), pp. 167–226.

Computation

A. A. Bennett, W. E. Milne and H. Bateman. *Numerical Integration of Differential Equations*, Bulletin 92, National Research Council, Washington, 1933.

Probability and Statistics

H. L. Rietz, *Mathematical Statistics*. Chicago, Open Court, 1927.

G. Udny Yale and M. G. Kendall. *An Introduction to the Theory of Statistics*, London, Griffin, 1937.

R. A. Fisher. *Statistical Methods for Research Workers*, London, Oliver and Boyd, various editions.

J. Neyman. *Lectures and Conferences on Mathematical Statistics*, Washington, Graduate School of the U. S. Department of Agriculture, about 1938.

S. S. Wilks. *Statistical Inference*, Ann Arbor, Edwards Bros., 1937.

T. C. Fry. *Probability and Its Engineering Uses*, New York, Van Nostrand, 1928.

W. A. Shewhart. *Economic Control of Quality of Manufactured Products*, New York, Van Nostrand, 1931.

Cryptography

L. Sacco. *Manuale di Crittographia*, Rome, 1936.

L. S. Hill. *Concerning Certain Linear Transformation Apparatus of Cryptography*, American Mathematical Monthly, vol. 38 (1931), pp. 135–154.

References in other fields will be suggested later.

THE JUNE MEETING IN SEATTLE

The three hundred seventieth meeting of the American Mathematical Society was held at the University of Washington in Seattle, Washington, on Thursday, June 20, 1940, in conjunction with the meetings of the American Association for the Advancement of Science. The attendance was about fifty-two including the following thirty-nine members of the Society:

H. M. Bacon, C. R. Ballantine, J. P. Ballantine, E. F. Beckenbach, K. E. Benson, Z. W. Birnbaum, H. F. Blichfeldt, A. F. Carpenter, Sr. Maria Corona, C. M. Cramlet, C. H. Dowker, T. C. Doyle, K. S. Ghent, H. E. Goheen, F. L. Griffin, Israel Halperin, I. E. Highberg, Ralph Hull, C. F. Luther, L. H. McFarlan, W. E. Milne, C. W. Moran, R. E. Moritz, E. J. Moulton, A. F. Moursund, Hermance Mullemeister, L. I. Neikirk, John von Neumann, E. W. Pehrson, E. D. Pepper, T. S. Peterson, T. M. Putnam, J. H. Raymond, I. S. Sokolnikoff, P. M. Swingle, A. H. Taub, G. A. Whetstone, R. M. Winger, H. S. Zuckerman.

During the morning session Professor John von Neumann gave an address entitled *On operator rings and dimension.*

The titles and cross references to abstracts of papers read at the meeting are given below. Papers whose abstract numbers are followed by the letter *t* were read by title.

1. J. P. Ballantine: *Proof of Poincaré's geometric ring theorem.* (Abstract 46-7-375.)

2. E. F. Beckenbach: *An integral analogue of Laplace's equation.* (Abstract 46-7-376.)

3. P. M. Swingle: *Non-transfinite connected sets.* (Abstract 46-7-389.)

4. T. S. Peterson: *The Fredholm minors for Goursat's kernel.* (Abstract 46-7-386.)

5. Ralph Hull: *The structure of certain Fuchsian groups.* (Abstract 46-7-383.)

6. H. E. Goheen: *On the primitive groups of classes* $4p$ *and* $5p$. (Abstract 46-7-380.)

7. A. F. Moursund: *A note on Gibbs' phenomenon.* (Abstract 46-7-385.)

8. G. A. Whetstone and C. M. Cramlet: *On the invariance of the passivity conditions of systems of partial differential equations.* (Abstract 46-9-400.)

9. C. W. Vickery: *On cyclically invariant graduation.* II. (Abstract 46-7-390-*t*.)

10. H. S. Wall: *A continued fraction related to partition formulas.* (Abstract 46-7-392-*t*.)

11. C. W. Vickery: *On systems of Fourier coefficients.* (Abstract 46-7-391-*t*.)

12. R. S. Burington and J. M. Dobbie: *A new family of wing profiles.* (Abstract 46-7-377-*t*.)

13. P. M. Swingle: *Local connectedness and biconnected sets.* (Abstract 46-7-388-*t*.)

14. E. L. Dodd: *The probabilities for certain inequalities among the values of a moving average of chance variables.* (Abstract 46-7-379-*t*.)

15. Y. K. Wong: *On biorthogonal matrices.* (Abstract 46-7-393-*t*.)

16. L. B. Hedge: *Moment problem for a bounded region.* (Abstract 46-7-382-*t*.)

17. R. M. Robinson: *On the approximation of irrational numbers by fractions with odd or even terms.* (Abstract 46-7-387-*t*.)

T. M. PUTNAM,
Associate Secretary

BOOK REVIEWS

Einführung in die algebraische Geometrie. By B. L. van der Waerden. (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 51.) Berlin, Springer, 1939. 247 pp.

During the past 15 years the branch of mathematics known as algebraic geometry has experienced a considerable revival of interest. This has been due largely to the introduction into this field of methods and results of the modern developments in topology and algebra. Professor van der Waerden has been influential in both aspects of this revival, particularly on the algebraic side. In a series of articles, most of which appeared under the general title "Zur algebraischen Geometrie," he has developed a basic technique for the investigation of the algebraic properties of loci defined by polynomial equations. *Einführung in die algebraische Geometrie* is mainly a systematic presentation of some of the principles established in these articles, together with applications to classical problems of algebraic geometry.

The first three chapters of the book provide an introduction to the general theory which follows. There is first defined the important notion of a space (projective or affine) over a field K. The space is defined in the usual way by means of coordinates, these being elements of K. For the benefit of later developments, K is assumed to be algebraically closed and of characteristic zero, but it is otherwise arbitrary. The remainder of the first chapter is taken up with the derivation of the familiar properties of projective spaces; among the topics discussed are duality, projective transformations and their classification, Plücker coordinates for linear subspaces, correlations, null-systems and linear complexes, and some properties of hyperquadrics and of cubic curves in 3-space.

In the second chapter are introduced the basic algebraic concepts which are used throughout the rest of the book. An algebraic function of the variables $u_1, \cdots, u_n$ is defined as an element of an algebraic extension of the field $K(u_1, \cdots, u_n)$. The properties of these functions are then developed, first in general and then in the case when K is the complex field. The familiar fractional power series expansion of a function of one variable in the complex case is shown to have an exact analogue in a formal power series expansion in the general case. These formal power series play an important part in the discussion of plane algebraic curves, which is the topic of Chapter 3. An algebraic curve in the projective plane (over K) is defined in the customary manner, and the multiplicity of a point of intersection of two curves

is determined by factoring a certain resultant connected with the curves; Bezout's theorem follows at once. The formal power series are used to introduce the notion of a branch of a curve; once this has been done the classification of singularities, their reduction by quadratic transformations and the derivation of Plücker's formulas follow quickly. A discussion of cubics and the point groups on them is also included.

Chapters 4 to 8 contain the basic theory of algebraic varieties. (We have translated "Mannigfaltigkeit" as "variety" rather than "manifold" for the latter word has a different meaning in topology.) An algebraic variety is the set of all points of a projective n-space whose coordinates satisfy a set (finite or infinite) of homogeneous polynomial equations. The most important varieties are the irreducible ones; those which cannot be expressed as the logical sum of two or more varieties. With each irreducible variety is associated a nonnegative integer d called its *dimension*. At this point the author pauses to give a precise meaning to that much abused term "general point." A set of $n+1$ elements $(\xi_0, \cdots, \xi_n)$ of some extension of K is called a general point of a variety in n-space if a necessary and sufficient condition that $f(\xi_0, \cdots, \xi_n)=0$, f being a homogeneous polynomial with coefficients in K, is that $f(a_0, \cdots, a_n)=0$ for every point $(a_0, \cdots, a_n)$ of the variety. It is shown that a variety is irreducible if and only if it has a general point. The extended use of the word "point," implied by the above definition, to include $(n+1)$-tuples of an extension of K, is continued throughout the rest of the book. One such point η is said to be a *specialization* of another ξ if every homogeneous polynomial equation satisfied by ξ is also satisfied by η. The notions of general point and specialization are fundamental in the discussion of varieties. Chapter 4 closes with a constructive method of finding the irreducible components of a variety and a proof that every variety over the complex field is a topological complex.

Algebraic correspondences are considered in the fifth chapter. An algebraic correspondence between two varieties M and N is defined as a variety in the product space of M and N. If this variety is irreducible, it has a dimension q. It is shown that in this case to a general point of M (or N) corresponds an irreducible variety of N (M), of dimension b (d); furthermore, if M (N) is of dimension a (c), then $q=a+b=c+d$. The usefulness of this result is demonstrated by applications to the intersections of varieties with linear subspaces of the containing space, to the proof of the existence of 27 lines on a nonsingular cubic surface, and to the construction and investigation of a coordinate system for the d-dimensional varieties of an n-space. In

Chapter 6 the properties of correspondences are combined with the concept of specialization of points to answer the important question as to how the multiplicities of the solutions of an algebro-geometric problem are to be counted. The results are then applied to the investigation of tangent spaces of a variety and the intersection of a variety with hypersurfaces of its containing space.

Chapter 7 treats the general theory of linear systems of varieties of dimension $d-1$ on a variety of dimension d. The elementary properties of these systems, their relation to rational transformations, the behavior of a simple point of the variety under such a transformation, and the Bertini theorems are discussed, much use being made of the general points of the varieties involved. There is also a short discussion of divisor classes, their addition, and complete systems. In the next chapter these concepts are applied to the special case of linear series of point groups on a curve. The usual treatment is given here, commencing with Noether's theorem and leading up to the theorem of Riemann-Roch. The generalization of Noether's theorem for hypersurfaces is also given, together with some applications to space quartics. The last chapter is devoted to an account of Enriques' analysis of neighboring points on a branch of a plane curve, and an examination of their behavior under quadratic transformations.

The most striking difference between the classical introduction to algebraic geometry and the one outlined above is the latter's essential avoidance of all notions of continuity. It is this adherence to purely algebraic processes that enables one to replace the complex numbers of the classical theory by the elements of the arbitrary field K. In carrying over the classical proofs to the more general case it is necessary to have substitutes for the limiting processes which often occurred in these proofs; these substitutes are found in the use of formal power series and the specialization of general points. By applying these processes the known results can be formulated and proved in a way that shows their essentially algebraic nature and also avoids the loose reasoning that has characterized certain branches of algebraic geometry in the past.

In presenting this material van der Waerden has attempted to eliminate all superfluous algebra, in particular the use of ideal theory. The only non-elementary knowledge required of the reader is an acquaintance with the properties of algebraic and transcendental adjunctions of fields, the general elimination theory, and those properties of polynomials centering around Hilbert's basis theorem. By thus avoiding the more intricate algebraic ideas the author is able to keep the geometric background of his arguments in sight at all times, thus

avoiding a tendency to over-abstraction which is apt to show itself in this type of work. The way the material is arranged also tends to emphasize the geometry—as each new principle is developed its use is illustrated by one or more applications to geometric problems, and in most cases some additional exercises are provided for the reader.

In his preface the author states that the book was written to provide the reader with the background necessary for the study of the deeper parts of algebraic geometry, especially the theory of surfaces. This requirement is perhaps a little vague, as the word "deep" may mean different things to different people, but by almost any criterion it may be said to have been well carried out. As noted above, the book contains a minimum of algebraic complications, and it certainly gives a good account of the basic notions of the subject. There are a few additional topics which the reviewer would have liked to see discussed, particularly the algebraic function field associated with an irreducible variety and its invariance under birational transformations. This is closely related to the notion of a general point and could easily have been introduced in Chapter 4. However, this is merely a detail; on the whole, we believe that the author has made a very good selection of the material at hand.

Technically the book lives up to the high standard we have been led to expect of its author and publisher. A few misprints and incorrect statements were noticed, but these are all of a trivial nature and can easily be detected and corrected by any conscientious reader. Probably the most serious defect is the lack of an index; this is particularly to be deplored in a book which is to serve as an introduction to a relatively unfamiliar branch of mathematics.

We recommend this book to the attention of every mathematician who is interested in either algebra or geometry, and particularly to those who believe that algebraic geometry is still a backward and unrigorous branch of mathematics. They will find here a clear, systematic exposition of an important new mathematical development, one which will undoubtedly have great influence in enlarging the interest in this fascinating field.

R. J. WALKER

The Decline of Mechanism in Modern Physics. By A. d'Abro. New York, Van Nostrand, 1939. 10+982 pp.

The author of this book is already known as a successful popular writer on science (cf. this Bulletin, vol. 34 (1928), p. 789, for review by T. C. Benton of *The Evolution of Scientific Thought from Newton to Einstein*, Boni and Liveright, 1927).

In brief, the book is good. The aim is "to give a semipopular presentation of the progressive growth of the new ideas, starting from the most elementary notions and giving due consideration to the mathematical development without which the new theories could never have been constructed." This aim is admirably fulfilled in three steps: Part I, "General Considerations" (114 pp.); Part II, "Physical Theories of the Classical Period" (312 pp.); Part III, "The Quantum Theory" (535 pp.). The three steps are progressively more technical.

The criticisms offered below are criticisms of detail. On the execution of his scheme, planned on such generous lines, the author is to be congratulated. It is to be hoped that potential readers will not be deterred by the formidable bulk of the book. Condensation is not desirable in a semipopular book, and there is advantage, rather than disadvantage, in a certain amount of repetition for emphasis.

The general scope of the book is indicated by the following samples of chapter headings (there are 41 chapters altogether): Assumptions in Science, The Significance of Theoretical Physics, Mechanistic Theories, Field Theories, Phenomenological Theories, Analytical Mechanics, Minimal Principles and Principles of Action, Thermodynamics, Planck's Original Quantum Theory, Bohr's Atom, De Broglie's Wave Mechanics, Heisenberg's Uncertainty Principle, Dirac's Theory of the Electron, The New Statistics.

The emphasis throughout is on theoretical physics rather than experimental. A novel feature is a popular presentation of pure mathematics, including functions of real variables, complex variables, elliptic functions, automorphic functions, groups of substitutions, differential equations, with a chapter on "The Controversies on the Nature of Mathematics."

A book of such length and of this character would be intolerable if dull or badly written. It is neither. It has a few dull patches but they are balanced by portions in which the interest is skilfully worked up. The style flows easily—perhaps a little too dignified—but science seldom smiles (in print).

Now for some criticisms. Before warming to his subject proper, the author had to (i) proclaim a new scientific era, (ii) kick the ancient Greeks. It is right and proper that the reader should know that this is an age of scientific revolution and that the Greeks were not the last word. But these things are so obvious that the information should be conveyed adroitly and, in the case of the Greeks, chivalrously, seeing that they are not in a position to answer back. The title of the book is not adroitly chosen. How many potential readers of a semipopular book know in what sense "mechanism" is to be understood?

Is it the insides of a watch? Or what? The author says what it is with a noble attempt to be clear and consistent, but flounders in the face of the impossible. One cannot tie words down like this. Half way through the book he seems to realize that all is not well: " . . . the meaning of the word mechanism needs revision, for today, since mechanics has been revised, there is no reason to associate mechanism with a form of mechanics which is recognized as a mere approximation." To save himself from intolerable confusion, the reviewer intends to continue to regard the insides of his watch as mechanism, and to think of the book under review as "Modern Physical Thought and the Quantum Theory."

As for the Greeks, we bare our teeth and rush to their defense on reading the following: "These qualities of courage, sincerity and modesty, which the Greeks seem to have lacked, prevented them from creating a science." We pass the lack of modesty (they were not alone), and even the courage and sincerity (though we have heard to the contrary), but on the point that they did not create a science, a stand must be made. Not only did the Greeks create *a* science, they created Science, in the sense in which it is understood in this book—namely deductive science. Any primitive people can collect data and record it: but that is not science. The dangerous and fascinating idea that nature is amenable to reason—that, for example, from a small number of axioms a geometry can be created which will not crack under application to astronomical observations or the surveying of fields—that courageous (if immodest) idea we owe to the Greeks. If not, to whom? Our forefathers in mediaeval Europe did not get it without the Greek contact.

Poor Aristotle comes in for special punishment, to quote: "We are told that a body dropped from the mast of a moving ship falls behind the mast. Obviously the experiment was not performed." Where does Mr. d'Abro think it would fall? It seems to the reviewer that a climb to the masthead would be an unnecessary exertion in order to verify such an obvious truth. Indeed Aristotle seems to have stuck to facts more than Galileo. He appreciated that terrestrial motions are conditioned by fluid resistance so that a terminal velocity (motus naturalis) cannot be exceeded, and the fact that Galileo succeeded where Aristotle failed might be ascribed in part to the fact that Galileo had the nerve to neglect experimental evidence and talk about motion in vacuo as if it existed physically. (See Nature, vol. 92 (1914), pp. 584, 606.)

In conclusion let it be repeated that the above are criticisms of detail. When he gets down to business the author proves himself a

master of lucid exposition, and the book is to be warmly recommended to physicists and mathematicians and to those members of the public who are interested in getting a general view of modern theoretical physics. The book is well turned out, shows only a few trivial misprints (the omission of "h" from "psychological" in the table of contents is the most shocking), and has a much fuller index than those found in other books of this sort (if there are other books of this sort).

J. L. SYNGE

Colloque Consacré à la Théorie des Probabilités. Edited by M. Fréchet and E. Borel. (Actualités Scientifiques et Industrielles, nos. 734–740.) Paris, Hermann.

The proceedings of this colloquium are published in eight small volumes. They comprise an excellent collection of articles which would be an extremely valuable addition to the library of anyone interested in the theory of probability. Although very little of the material is of a purely expository nature, these volumes furnish a rather complete picture of the modern developments of this theory. The following is an outline of the contents of the various conferences.

Volume I. *Conférences d'Introduction et d'Initiation.* 1938

This volume contains two introductory addresses, 1. *Introduction*, by R. Wavre, 51 pages, and 2. *Allocution*, by M. Fréchet. These addresses are followed by:

3. *Les principaux courants dans l'évolution récente des recherches sur le calcul des probabilités*, by M. Fréchet. The author outlines the contributions, trends, and methods of the modern theory of probability. This paper constitutes only the first part of Fréchet's discussion. The remainder appears in Volume II.

4. *Promenade au hasard dans un réseau de rues*, by G. Pólya. The author considers certain probability problems leading to linear partial difference equations of the second order. Limiting cases of these problems admit of physical interpretations. In the limit the difference equations become differential equations. Moreover the solutions of these differential equations with suitable boundary conditions give asymptotic values for the solutions of the difference equations. The author discusses a promenade along a street of infinite length in which the direction of promenade is settled by the tossing of a coin at the end of each block. The corresponding physical problem is the diffusion of a salt solution in a tube. Furthermore the motion of rocks

in a river is related to the above promenade. An n-dimension promenade is also discussed. The reasoning is heuristic and the paper is suggestive of further research.

5. *Wahrscheinlichkeitsaussagen in der Quantentheorie der Wellenfelder*, by W. Heisenberg. The author states that a fundamental problem in modern physics is to ascertain at what points, in the description of natural phenomena, statistical considerations play a deciding role. As an example he considers the emission of electrons from a point source, the electrons being permitted to pass through two slits and strike a nearby plate. This may be regarded as a statistical phenomenon and hence we should expect that the probability of reaching a given point of the plate through one slit plus the probability of reaching this point through the other slit would be equal to the probability of reaching the point through one slit or the other. But this contradicts the fact that the problem can be regarded as a wave phenomenon and hence that there should be interference bands. The author mentions a number of other difficulties which have arisen in modern physics and suggests that the resolution of these difficulties may be accomplished by the introduction of new statistical considerations.

Volume II. *Les Fondements du Calcul des Probabilités*. 1938. 99 pp.

This volume constitutes a fascinating debate on the foundations of the theory of probability. A variety of points of view (not only of the participants of the congress but also of those who were unable to attend) are presented and criticized. Frequently answers to criticisms appear. This volume furnishes excellent material for the study of the philosophical aspects of the theory of probability.

1. *Sur la définition des variables éventuelles*, by P. Cantelli. Since Cantelli was unable to attend the congress only an abstract of his paper appears in this series.

2. *Sur les axiomatiques du calcul des probabilités et leurs relations avec les expériences*, by W. Feller. The first part of this paper contains a general discussion of the foundations of the theory of probability. The second part is devoted to an exposition and critique of E. Tornier's theory of probability. The exposition is much clearer than that found in Tornier's book (*Wahrscheinlichkeitsrechnung und algemeine Integrationstheorie*). Feller criticizes Tornier's objections to the complete additivity in Kolmogoroff's system. He notes Tornier's failure to recognize that his system is a special case of that of Kolmogoroff and that his use of the continuity axiom is precisely that of Kolmogoroff. To these criticisms I should like to add the following.

In order to avoid the difficulties of the Regellosigkeitsprinzip of von Mises, Tornier constructs a theory which is oversimplified to such an extent that it presents a much less accurate picture of the physical universe. This sacrifice is unnecessary. See for example the paper by Wald or the modified systems described by Fréchet in this volume.

3. *Exposé et discussion de quelques recherches récentes sur les fondements du calcul des probabilités*, by M. Fréchet. The author starts with the classical concept of probability of Laplace and Poincaré. He recalls the familiar objections to this point of view and indicates how these objections might be overcome. Next he presents the theory of von Mises and the objections which have been raised against that theory. He also adds objections of his own. Then he describes the modifications of the von Mises theory by Popper, Reichenbach, Copeland, Wald, and Ville. He argues that although these modifications avoid the serious logical difficulties of the von Mises theory, they are open to a new objection, that is, that they are less natural and do not possess the strong intuitional appeal of the original theory. The author then turns to a point of view which he finds more acceptable and which has been described by Neyman as the modernized classical theory. This point of view is based on the postulates of Kolomogoroff and the modern theory of measure. Whether or not one agrees with the author he cannot fail to recognize that this article contains an excellent exposition of the various points of view and a clear presentation of the difficulties which attend them.

4. *Quelques remarques sur les fondements du calcul des probabilités*, by R. de Misès. The author gives a brief exposition of his own frequency theory of probability and points out the analogy between it and the theory of mechanics. He accepts the refinements contributed by Wald, Copeland, and other authors but states that these refinements do not alter essentially the fundamental character of his theory. As to Fréchet's contention that these refinements tend to rob his theory of its intuitive appeal, he points out that this is the case with any mathematical concept but that such refinements are necessary for a firmer foundation. He remarks that the modernized classical theory of probability is insufficient for the solution of applied problems and that without the frequency theory the modern methods such as chains of probabilities, convergence in probability, and arbitrary functions become merely problems in algebra and point set theory.

5. *Fréquence et probabilité*, by J. F. Steffensen. Steffensen upholds the frequency theory of probability as being more natural and more easily applicable in problems of statistics. He bases his discussion of

this theory on two hypotheses: first, that every possible sequence of successes and failures should be taken into consideration and second that if two fortuitous sequences are continued sufficiently far they will eventually differ. From these hypotheses he concludes that no sequence determined a priori by mathematical law can occur. He states that the probability approaches 1 that the relative frequency will differ from the probability by an arbitrarily small amount and gives a new proof of this fact. He then shows that the additive and multiplicative laws of probability can be similarly treated.

6. *Die Widerspruchsfreiheit des Kollektivbegriffes*, by A. Wald. A Kollektiv is an infinite sequence of elements taken from a given label space. Probabilities are associated with subsets of this label space and these probabilities are defined as limits of success ratios with respect to the Kollektiv. These limits must be invariant under the operation of selection. Thus far Wald agrees with von Mises. However instead of the absolute Kollektiv of von Mises, Wald's Kollektiv is relative to a given family of subsets of the label space and to a given system of selections. That is, probabilities are required to exist only for subsets belonging to the family, and invariance is required only with respect to selections of the given system. He shows that if the family of sets constitutes a denumerable field including the label space and if the system of selections is also denumerable, then there corresponds a continuum of Kollektivs. Thus the system can contain all selections which can be defined within a given symbolic logic. He shows further that if the denumerable field and the denumerable system are constructively defined, then there exist Kollektivs which can be constructively defined. The remainder of the paper is devoted to showing the relation of his system to those of Reichenbach, Popper, and Copeland and to answering objections raised against the frequency theory.

Volume III. *Les Sommes et les Fonctions de Variables Aléatoires*. 1938. 68 pp.

1. *Sur un nouveau théorème-limit de la théorie des probabilités*, by H. Cramér. The author considers the probability that $Z_1+Z_2+\cdots+Z_n \leqq x\cdot\sigma\cdot(n)^{1/2}$ where $Z_1, Z_2, \cdots$ are independent fortuitous variables having a common distribution function and such that the expected value of each is 0 and the expected value of the square of each is σ^2. It is well known that as n becomes infinite this probability tends to the probability integral if x is constant with respect to n. However the author is interested in this limit when x is allowed to become infinite with n. By means of a certain transformation using

characteristic (or moment generating) functions this problem can be studied with the aid of Liapounoff's theorem. He considers the case in which x is of order $n^{1/2}/\log n$ and the case in which x is of order $n^{1/2}$. He obtains theorems of Khintchine, Lévy, and Smirnoff as special cases of his results. In the latter part of the paper he applies his method to the study of homogeneous stochastic processes.

2. *L'arithmétique des lois de probabilités et les produits finis de lois de Poisson*, by P. Lévy. This paper deals with the field consisting of all probability laws. A probability law can be defined either by means of a distribution function or by means of the corresponding characteristic function. If two independent fortuitous variables have respectively the probability laws L and L', then the sum of the two variables has the law denoted by the product LL'. The characteristic function corresponding to the law LL' is the product of the characteristic functions of the laws L and L'. The set of all probability laws constitutes a field in which a unity is a law corresponding to a fortuitous variable which can take on only a single value. This memoir is a study of the decompositions of the elements of the field into factors.

3. *Généralisation des théorèmes de limite classiques*, by R. de Misès. The author studies the asymptotic distributions of certain functions of n fortuitous variables when n is allowed to become infinite. As an aid in this study he considers functionals of the corresponding distributions. For example the moments of a given distribution are linear functionals of the corresponding distribution function. These functionals are defined over convex sets of distribution functions and derivatives are defined at points of these sets in such a manner as to permit Taylor's developments of the functionals.

Volume IV. *Le Principe Ergodique et les Probabilités en Chaîne*. 1938. 66 pp.

1. *Statistische Probleme und Ergebnisse in der klassischen Mechanik*, by E. Hopf. The equations of motion of a mechanical system in general define a one-parameter group of measure preserving transformations T_t of the phase space Ω into itself (the parameter t being the time). A transformation is said to be metrically transitive provided any subset of Ω which is transformed into itself has either the measure 0 or else the same measure as Ω. The condition of metric transitivity is necessary and sufficient that the limit average time that a point spends in a subset A of Ω is equal to the measure of A divided by the measure of Ω (Birkhoff's ergodic theorem). If the system is metrically transitive but fails to possess a characteristic frequency,

then every subset of Ω becomes uniformly distributed over Ω as the time is indefinitely increased (mixture theorem). The rest of the paper is devoted to the discussion of the weaker condition, regional transitivity.

2. *Les fluctuations (changement aléatoires du nombre de points ou d'objets dans un compartiment)*, by B. Hostinsky. The author considers three problems. The first concerns the distribution of N objects into n compartments. He computes the probability that x objects will be placed in a given compartment assuming that $1/n$ is the probability that a given object will be placed in that compartment. The limiting case in which N and n become infinite gives the Poisson law. The second problem concerns two urns with r balls in each. There are r white and r black balls, but otherwise the compositions of the urns are arbitrary. From each urn a ball is drawn at random and then placed in the urn from which the other was drawn. The author considers the limiting probability of a given composition as the number of such exchanges becomes infinite. This problem is analogous to the mixture of a colored liquid and a non-colored liquid in two containers which are connected. The third problem concerns two urns containing a total of three balls of the same color. Probabilities are given for the transference of a ball from one urn to the other and for failure to make such transference. These probabilities depend on the compositions of the urns. The limiting probabilities for various compositions are again computed. This problem is analogous to the exchange of compressible fluids or gases between two connected containers.

3. *Théorie générale des chaînes à liaisons complètes*, by O. Onicescu. The author considers a matrix composed of mutually exclusive fortuitous events (or states) $a_1, a_2, \cdots, a_m$ and the probabilities $x_1, x_2, \cdots, x_m$ corresponding to these events. Such a matrix can be transformed into a new matrix with new probabilities $y_i = f_{ik}(x_1, x_2, \cdots, x_m)$ for the given events, k being determined by the event a_k which does occur. Such a system of transformations constitutes a chain. If the matrix f_{ik} is independent of the probabilities, the transformation is completely fortuitous and the system constitutes a chain with complete liaisons. A chain is normal if the matrix f_{ik} possesses the ergodic property enjoyed by the Markoff chains. Such a matrix transforms the region $0 \leqq x_i \leqq 1$ into a closed region totally interior to this, and the limiting region as the number of transformations becomes infinite is a single point. The author considers also the continuous case in which the number of states is infinite. He cites as an example the equation of the flow of heat.

4. *Quelques problèmes nouveaux de la théorie des chaînes de Markoff*, by V. Romanovsky. The author considers a simple Markoff chain and a second chain such that there is a definite conditional probability of a given state on a given trial for every possible state of the corresponding trial of the first chain. The second chain is in general multiple and nonhomogeneous. A method is given for computing moments and the limiting correlation coefficient with respect to the chains. The author next studies cyclic chains. These chains may contain several cycles which occur fortuitously. A method of matrices is used to simplify the study of these chains. The latter part of the paper is devoted to methods of estimating by statistical means the law underlying a Markoff chain.

Volume V. *Les Fonctions Aléatoires*. 1938. 73 pp.

1. *Équations différentielles stochastiques*, by S. Bernstein. This paper is concerned with a difference equation of the form $\Delta y = A\Delta t + f(\Delta t)^{1/2}$ where A and f depend on y, t, Δt, and a fortuitous variable α_t and where the expected value of f is 0. Sufficient conditions are given for the existence of a limiting distribution function for y as the differences Δt approach 0. Bernstein shows that these conditions cannot be materially weakened. If $f \equiv 0$ and A is not a function of α_t, the problem reduces to the solution of the differential equation $dy/dt = A(y, t)$. Moreover if $(\Delta t)^{1/2}$ is replaced by Δt in the second term of the right-hand member of the difference equation and A is independent of α_t then y satisfies the equation $dy/dt = A(y, t)$ and the solution does not possess a fortuitous character. The method of this paper can be extended to systems of difference equations of the above form.

2. *Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires stationnaires en composantes*, by E. Slutsky. A fortuitous function y_t is assumed to be stationary, normalized, stochastically continuous, and subject to certain other minor conditions. The correlation $\rho(\tau)$ between y_t and $y_{t+\tau}$ can be decomposed into a constant plus a component with a discrete spectrum plus a component with a continuous spectrum. The portion with the discrete spectrum is an almost periodic function developable in a Fourier series. The function y_t admits of a decomposition into three non-correlated components the first of which is the mean value of y_t the second of which consists of a Fourier series closely related to the corresponding series for $\rho(\tau)$, and the third of which has a spectrum continuous in the mean. If $\rho(\tau)$ is almost periodic, then the probability is 1 that the Fourier series for y_t converges to y_t for almost every t.

3. *La théorie et les applications des fonctions indépendantes au sens stochastique*, by H. Steinhaus. If $f(t)$ is defined in the interval 0, 1, then the corresponding distribution function $F(x)$ is defined as the measure of the set of points on which $f(t)<x$ and is interpreted as the probability of this latter inequality. By suitable modifications this definition can be extended to any finite interval and also to the infinite interval. Independence is defined as independence in the probability sense. The functions $f(t)$, $g(t)$ defining the space filling curve of Peano furnish an example of independent functions. Such independent functions are an aid to calculation in many probability problems. The author applies this method to the probability limit theorem, the Maxwell distribution law, and to Brownian movements.

Volume VI. *Conceptions Diverses*. 1938. 57 pp.

1. *Sur la conception d'équivalence partielle*, by B. de Finetti. Finetti bases his theory of probability on the assumption that the laws of total and composite probabilities are valid and on a concept which he calls the equivalence of events. Events are said to be equivalent provided the probability that n of them will occur depends only on n and not on the particular set of n events chosen. He derives a formula (not the classical and highly dubious application of Bayes' principle) by means of which one can estimate the probability that one of the equivalent events will occur on the basis of an observed success ratio. Except for a very special case the formula shows that this probability must be close to the success ratio if the number of trials is large and that the events must be independent. The formula can be extended to groups of events where events in given groups are equivalent. This paper also contains a philosophical discussion concerning the relations between probability, causality, and induction.

2. *Sur la loi des grands nombres dans l'espace fonctionnel*, by V. Glivenko. Suppose that a fortuitous event has a continuous probability density defined in a closed interval. A histogram represents the observed frequency distribution in n trials of the event. The author shows that the probability is 1 that the maximum difference between the histogram and the probability density approaches 0 as the number of trials becomes infinite. This result is based on certain theorems which the author proves concerning moments of fortuitous functions. In particular he proves a generalization of the Bienaymé-Tchebycheff inequality and the strong law of large numbers in function space. The article though brief is complete and little previous knowledge of the theory of probability is assumed.

3. *Les fonctions aléatoires (lois de répartition dans un espace fonc-*

tionnel) et leurs applications, by A. Kolmogoroff. This paper was not completed at the time these volumes were published. It will appear in the collection *Exposés d'Analyse Générale* published under the direction of M. Fréchet.

4. *L'estimation statistique, traitée comme un problème classique de probabilité*, by J. Neyman. Let $X_1, X_2, \cdots, X_n$ be a system of fortuitous variables depending on the unknown parameters $\theta_1, \theta_2, \cdots, \theta_s$. An experiment determines the values of the fortuitous variables, that is, a point of the euclidean space R_n. Let a set function of points be defined such that to every point of R_n is assigned a set of points of the real axis. If this function is properly determined, then when $X_1, X_2, \cdots, X_n$ is substituted for the point of R_n there will be determined a probability that a given point θ_1 of the real axis will be included in such a set, θ_1 being one of the parameters. Let the function be further restricted so that this probability is a preassigned number α (the coefficient of confidence) for all possible values of the parameters. It follows that when an experiment determines a point of R_n the probability is α that the parameter θ_1 will be found in the corresponding set. Suppose that the number of parameters is 1 and that the above function is further restricted so that the probability α is maximized (maximum likelihood). The sets (functions of points of R_n) are then intervals (confidence intervals) and are uniquely determined.

Volume VII. *La Statistique Mathématique*. 1938. 57 pp.

1. *Certain coefficients of regression or trend associated with largest likelihood*, by E. Dodd. A pair of fortuitous variables x, y is assumed to be subject to a probability distribution which is known except for the values of certain parameters. A set of observations is made determining n values of each variable. The parameters are then determined by maximizing the likelihood of obtaining the values which were actually obtained. The author considers two examples and in each case the parameters are determined as certain means closely related to regression coefficients. The author then considers a general concept of mean and a corresponding theorem concerning maximum likelihood. However the conditions of this theorem are such as to permit an arbitrary assignment of value to the unknown parameter.

2. *Critique de la corrélation au point de vue des probabilités*, by C. Jordan. The author considers the correlation between two fortuitous variables u and v. A measure called the intensity of correlation is defined in terms of the discrepancies between certain conditional probabilities and corresponding unconditional probabilities. This

measure is equal to 0 if and only if the variables are independent and it attains its maximum 1 if and only if the variables are completely dependent. Although the intensity of correlation is never negative and hence cannot distinguish between positive and negative correlation, the author gives a simple independent method of accomplishing this distinction. The intensity of correlation is closely related to two measures given respectively by Pearson and Tschuprow. Jordan also discusses the conventional correlation coefficient, two additional measures due to Pearson, and Pearson's correlation ratios. The intensity of correlation is the only one of these measures which possesses all of the above mentioned desirable properties.

3. *Sur la loi de Poisson, la loi de Charlier et les équations linéaires aux différences finies du premier ordre à coefficients constants*, by N. Obrechkoff. The first portion of this paper concerns a system of functions consisting of the Poisson function $(a^x/x!)e^{-a}$ and its differences. Any function of the system is the product of the Poisson function with the appropriate Charlier polynomial. The author gives sufficient conditions for the development of a given function in a series formed from this system. The second part of the paper deals with a pair of dependent fortuitous variables such that each has a Poisson distribution when the other is fixed. This condition determines the formulas for the distribution of the two variables and the unrestricted distribution of each of the variables. The last part of the paper concerns a certain type of linear nonhomogeneous difference equation. The author gives a formula for the asymptotic behavior of the solution.

A. H. COPELAND

Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung. By Karl Stumpff. Berlin, Springer, 1939. 172 pp.

This book is a sequel to the author's volume *Grundlagen und Methoden der Periodenforschung* (Berlin, Springer, 1937). It contains extensive tables for the carrying out of harmonic analysis of empirical curves based upon interpolation by means of trigonometric polynomials of proper degree through equally spaced ordinates. Tables are given for various numbers p of equal divisions. Tables are given for the proper sines and cosines by which to multiply the p ordinates in order to obtain the proper Fourier coefficients. These tables extend as far as $p=40$. They also include the best laborsaving arrangement for the ordinates. These tables are followed by the first thousand multiples of cosines and sines of various angles which occur for $p=8$, 12, 16, and 24, as well as the first hundred multiples of the cosines and sines of all angles of integral degrees in the first quadrant. These

tables enable one to avoid a great deal of the multiplication of the ordinates by the proper cosines, which is inherent in the calculation of the coefficients. Further tables include the conversion from polar to rectangular coordinates (useful in passing from the two Fourier coefficients corresponding to a particular frequency to its amplitude and phase), Darwin's scheme for a large number of ordinates ($p=121$), tables of squares, square roots, $(\sin \alpha)/\alpha$, and other tables of interest in harmonic analysis. The last quarter of the book is devoted to problems and examples in harmonic analysis, illustrating the use of the various preceding tables; also to problems in the analysis of hidden periodicities.

The present volume should prove very useful to anyone engaged in frequent harmonic analysis.

H. PORITSKY

Tables of Partitions. By Hansraj Gupta. Madras, Indian Mathematical Society, 1939. 5+81 pp.

This work contains two tables. The first table gives the number $P(n)$ of unrestricted partitions of n for $n \leqq 600$. The second table is a table of double entry giving the number (n, m) of partitions of n the least element of which is m for $n \leqq 300$. As is explained in the introduction, the second table was calculated by the recursion formula $(n, m) = (n-m, m)+(n+1, m+1)$ together with certain special properties of the symbol (n, m); a specimen calculation is given. The first table is a result of the second since $P(n) = (n+1, 1)$. Adequate checks have been applied to insure accuracy. The introduction also contains summaries of certain papers by G. N. Watson and D. H. Lehmer on partitions.

The Indian Mathematical Society and the University of the Punjab are to be commended for bearing the cost of publication of this work. The table of unrestricted partitions has previously been published in two parts in the Proceedings of the London Mathematical Society. This table has been used by Chowla to disprove one case of a result conjectured by Ramanujan. This work should be very valuable to anyone interested in the subject of partitions.

T. A. PIERCE

Elementary Mathematics from an Advanced Standpoint. Geometry. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, Macmillan, 1939. 9+214 pp., 141 figs.

Another volume of Klein's masterful lectures is available in Eng-

lish! This book—it should be found on the shelves of every mathematical library—will be read with pleasure and profit alike by the scholar, the student, and the teacher. It preserves, even in translation, all those qualities for which the lectures of Klein are so justly celebrated.

The first volume of Klein's *Elementarmathematik vom höheren Standpunkte aus* was translated also by Professors Hedrick and Noble (reviewed in this Bulletin, vol. 39 (1933), pp. 495–496), and this, the second, is similar in character and purpose: "In the foreground I shall place, let me say, the *encyclopedic ideal*—you will be offered a *survey of the entire field of geometry* into which you can arrange, as into a rigid frame, all the separate items of knowledge which you have acquired in the course of your study, in order to have them at hand when occasion to use them arises. Only afterward shall I emphasize that *interest in mathematical instruction* . . . " (see page 1). The emphasis is on *ideas*, as opposed to the details involved in their development, and on the *interrelations between different fields*.

Klein considers geometry from the point of view developed systematically in his *Erlanger Programm* of 1872. His book has a wide appeal in spite of the fact that the last statement might seem to indicate a theoretical treatment of interest only to the mathematical specialist. The explanation is found in the large number of applications to a wide variety of problems.

The book is divided into three parts: Part I, "The Simplest Geometric Manifolds"; Part II, "Geometric Transformations"; Part III, "Systematic Discussion of Geometry and Its Foundations." In Part I we find a discussion of all the classical geometric manifolds, that is, classes of objects which are investigated in geometry. Many of these manifolds (various types of vectors, tensors, and so on) are of interest to physicists. A detail is an explanation of the theory of Amsler's polar planimeter. We find in Part II a discussion of affine transformations, projective transformations, higher point transformations, and transformations with a change of space element. It is shown that the axonometric mapping of space on a plane used in descriptive geometry is an affine transformation with a vanishing determinant. Relief perspective and geographical maps are considered. There is an application of contact transformations to the theory of cog wheels. In the first section of Part III we find a classification of geometries according to their groups and a discussion of invariant theory. The second section of Part III contains the material of immediate interest to prospective teachers of high school geometry. Two methods of developing plane geometry, different from that followed by Euclid,

are outlined. One emphasizes motions; the other takes distance, angle, and congruence as the fundamental notions. Finally, there is a critical discussion of Euclid's *Elements*.

A reading of this book makes one aware of recent progress in mathematics. To Klein, writing in 1908, geometry was coordinate geometry. He makes only bare references to point sets as objects of geometric study and to Hilbert space. He gives only a brief discussion of homeomorphisms and even then only in euclidean 3-space. Geometric manifolds with the complete generality of our modern abstract spaces were then unknown. Metric spaces (in the sense of Hausdorff) had been introduced only in 1906, and topological spaces were not yet clearly defined. The types of geometry that now occupy the center of the stage were only beginning to emerge. Nevertheless, they too fit into the framework erected by Klein.

G. B. PRICE

Advanced Calculus. By Ivan S. Sokolnikoff. New York, McGraw-Hill, 1939. 10+446 pp.

In most American colleges there is an abrupt discontinuity in mathematical education at the end of a first course in calculus. The drop in enrollment due to the completion of required mathematics by hordes of engineering students makes a sudden change in the needs and capabilities of the students. This discontinuity is perhaps more apparent to outsiders than to mathematicians themselves. It is the reason for the widely held popular belief that all useful mathematics ends with the calculus, that Newton made the last discoveries in mathematics, and that mathematics is dead except for a few frills indulged in by a handful of eccentrics. Even the name "Advanced Calculus" encourages this misconception—as though it were just calculus over again, with a few added tricks to be sure, but no new ideas.

It is in a second course in calculus that the number of students with primary interest in mathematics is for the first time an appreciable percentage of the total. The problem of the relative emphasis on method, that is, proofs, versus utility, to meet the respective needs of mathematics majors and students in the applied field, is as perplexing as any problem in the whole of mathematical education. Some sort of compromise is usually necessary. It would be very useful to the teacher if textbooks of good craftmanship were available with a variety of methods of effecting such a compromise. Unfortunately, a number of the existing books in this field are (or at least have the bulk of) treatises, and are not entirely suitable for textbook use.

The reviewer, therefore, welcomes the contribution of a book, entirely suitable for classroom use, from a man who has personal inclinations toward applied mathematics and who has already made a definite contribution in this field with a textbook leaning toward applications, but who now presents a treatment with considerable emphasis on real variable theory.

After an introductory chapter on the concepts of limit and continuity, there are chapters on the notion of derivative, functions of several variables, definite integral, multiple and line integrals, infinite series, power series, improper integrals, and Fourier series. A final chapter in implicit functions is more in the nature of an appendix; the treatment of partial differentiation of implicit functions in an early chapter is only claimed to be heuristic.

By limiting the scope of the book to the central problems of the calculus, the author is able to include a reasonable amount of formal material and at the same time give the student an introduction to "ϵ, δ" arguments. An attempt is made to reduce these arguments to terms understandable by the average student by such devices as prefacing the actual proofs with heuristic treatments, and by the use of geometric counterexamples. Numerous exercises are also included.

The first chapter is evidently not intended to be taken too seriously; otherwise such a variety of new abstract ideas as Dedekind cuts, ϵ, δ definition of limit, Cauchy criterion for convergence, limit point, inferior and superior limits, uniform continuity, and so on, would overwhelm the average student just out of sophomore calculus. For such students the teacher might want to start with Chapter II and gradually refer back to the first chapter. The proofs in this opening chapter are incomplete, and many of the theorems are merely stated. For example, the proof of the Cauchy criterion for convergence is carried just to the point where one would ordinarily construct a cut; then the author says the remainder of the proof is too difficult and will be omitted. Such a procedure could have been avoided by merely postulating the existence of a least upper bound of any bounded set of real numbers.

Those who believe in being absolutely meticulous of statement in a book of this level will have a number of quarrels with the author. His definition of Riemann integral is inaccurate. Also, although he gives an ϵ, δ definition of limit of a function, he encourages the student to think in terms of the independent variable x approaching x_0 "by any conceivable sequence of steps," without ever making this idea precise. An extension of this vague idea to limits of functions of two independent variables has led the author to the erroneous

conclusion that the existence of the simultaneous limit always insures the existence of the iterated limits. This conclusion is, in turn, involved in the proof of the theorem on the interchangeability of order of partial differentiation. (Fortunately, this does not invalidate the proof, but it makes desirable further elucidation of some of the steps.) Finally, it is stated, without proof, that the mere existence of f_{xy} and f_{yx} is sufficient to insure their equality; but counterexamples to this statement are known.

If it is desirable to use the intuitive idea of different modes of approach of the independent variable to its limit, the notion could easily be put on firm ground by giving the Heine sequence definition of a limit and stating its equivalence to the Cauchy definition. That this equivalence is not as simple as appears on the surface is well known, for the proof is impossible without the axiom of choice.

The chapters on infinite series, power series, and Fourier series appealed to the reviewer as very well done. The chapter on Fourier series contains among other topics a simple account of the Dirichlet conditions for convergence, Parseval's equation, the complex form of Fourier series, the asymptotic behavior of the coefficients, and the differentiation and integration of Fourier series.

P. W. KETCHUM

Geometria Descrittiva: Lezioni Redatte per Uso Degli Studenti. By Enea Bortolotti. Padua, Cedam, 1939. 715 pp., 500 figs. (Mimeographed.)

The term descriptive geometry has a wider meaning in Europe than it has with us; this is especially true in Italy where the study of geometry is an important part in the curriculum of all students of mathematics. Moreover, the school programs include work in determinants and matrices, projective and analytic geometry, and the elements of the calculus. Students of mathematics in the universities analogous to our undergraduates, devote all their time to the subject, hence are taking three courses simultaneously. By the time they reach descriptive geometry they will have had sufficient training to allow a teacher or an author to assume an acquaintance with many fundamental concepts.

Textbooks are ordinarily not employed at all; books written on a specific subject are for supplementary reading, usually voluntary, and not controlled. The students are thrown on their own at an early stage, and many of them show a decided precosity. The books are written for the better students who really want to learn about the subject treated.

The book under review is divided into four parts; most of the

second can be read without the first, but the later parts make use of both. The first is devoted to descriptive geometry in the narrower sense and covers 243 pages. It starts with h, v orthogonal projections of figures bounded by straight lines and planes. Intuition is used freely, but each theorem is proved rigorously, including those on shades and shadows. Very few hints are given, but occasionally the reader is advised to go through the details of that particular construction. The second chapter of this part treats of perspectivity, both central and parallel, orthogonal and oblique. This is a helpful addition. So many books omit it in the elementary presentations, while more advanced treatments (sometimes in the same book) presuppose it. Now follows anaxometry. The discussion is so condensed that the reviewer frequently wondered whether a reader not having additional help could always follow the reasoning. The theorem of Pohlke is proved and frequently used. A commendable feature is the comparison of the respective advantages of the three methods developed in these three chapters. The closing chapter of this part gives a brief summary of a method used in topography, especially in mapping curves of equal level.

Thus far the subject matter has been almost entirely self-contained. All theorems used have been proved as needed. The second part, curves and surfaces, covers nearly half the volume; it constantly refers to proofs of theorems given in other Italian texts rather than spend time and space on them in the present work. It begins with plane curves, and provides an extensive treatment of plane differential geometry, including a discussion of singularities. It is carefully done, but the processes, applied to a complicated case would become so cumbersome as to be useless. This chapter is followed by a briefer one on space curves and developables. Thus far, no use has been made in this chapter of any of the earlier ones.

Now follows an extensive but condensed discussion of surfaces, from the standpoint of differential geometry; it includes normals, tangent planes, asymptotic lines, geodesics, lines of curvature, conjugate systems, and so on, with special emphasis on ruled surfaces and surfaces of revolution. Many names are used, as discoverers of theorems met, but there is no citation of sources. The mapping of one surface on another is discussed and applied in detail to various kinds of map drawing, in which one property or another is preserved, such as lengths, angles, areas, and so on. Here repeated use is made of the results of the preceding chapters.

A short chapter on graphical calculus and nomography is followed by one called "Complementi," in which details of the theory of linear

transformations in one, two, and three dimensions, as well as correlations in two and three dimensions are developed, including the fixed points and a considerable number of particular cases. This is followed by a more detailed study of algebraic curves and surfaces. In the plane, polarity is introduced and used to derive Plücker's numbers, with application to cubic curves. The configuration of the points of inflexion, the constant cross ratio of the tangents from a point on the curve, nodal and cuspidal forms are provided for. Space cubic curves, and both kinds of space quartics are treated briefly.

Even with the omission of many proofs, the development is so rapid and condensed that a reader must be alert and patient to get all the points of the argument. On the other hand, one who has mastered this volume will be in possession of a large part of the knowledge which goes under the generic name of geometry.

VIRGIL SNYDER

Some Integrals, Differential Equations and Series Related to the Modified Bessel Function of the First Kind. By A. H. Heatley. (University of Toronto Studies, Mathematical Series, no. 7.) Toronto, University Press, 1939. 32 pp.

Let $I_n(x)$ denote Bessel's function, and let $T(m, n)$ denote the integral over $0<t<\infty$ of the function $I_n(2at)t^{m-n}\exp(-p^2t^2)$. Differential equations are given for $T(m, n)$, for $T(m, n)\exp(-a^2/p^2)$, and for $T(m, n)\exp(-a^2/2p^2)$; and these lead to explicit formulas for $T(2n+1, n)$ and for $T(n, n)$. Then recursion formulas lead to evaluation of $T(m, n)$ for other pairs of values of m and n.

Power series expansions of $T(m, n)$ and $T(m, n)\exp(-a^2/p^2)$ are obtained. These results of Part I (pp. 7–21) are used in Part II (pp. 21–30) to evaluate an integral used by the author (Physical Reviews, vol. 52 (1937), pp. 235–238) in the Langmuir collector theory.

Part III (pp. 30–32) deals briefly with integrals of $x^{n+m}e^{x}I_n(x)$ and $x^{n+m}e^{-x}I_n(x)$.

R. P. AGNEW

Theory of Probability. By Harold Jeffreys. Oxford, Clarendon Press, 1939. 7+380 pp.

This book of Jeffreys is an outstanding addition to the relatively few substantial treatises of probability in English. It is in line with the author's *Statistical Inference.* At the start, it resembles Keynes' *A Treatise on Probability* in its subjective or psychical approach to probability. But it carries the implications of this approach to a great variety of problems arising in the physical sciences, in biology and in

economics. This subjective axiomatic approach to probability is exemplified also by a recent paper of B. O. Koopman, *The axioms and algebra of intuitive probability*, in the Annals of Mathematics, April, 1940.

Keynes, Jeffreys, and Koopman deal first with non-numerical probability. The probability of a proposition is an extension of logic by interpolation between the false or impossible proposition and the true or certain proposition. First, these subjective probabilities are assumed to be orderable. However, Keynes does not even require the order to be linear—he gives on page 39 a two-dimensional diagram showing several curves extending from the point of impossibility to the point of certainty. Points on different curves represent propositions with probabilities not comparable. But Jeffreys writes: "Axiom 1. Given p, q is either more or less probable than r, or both are equally probable; and no two of these alternatives can be true." After Theorem 2, Jeffreys sets up his "Convention 3" which associates certainty with the number 1. But he states that cases arise where ∞ is a more suitable choice for certainty. His Theorem 8 is: "Any probability can be expressed by a real number."

Jeffreys criticizes other conceptions or definitions of probability as follows: The "classical" definition of probability leads to nothing but combinatorial analysis. Probability is not introduced until human judgment assesses the elements to be equally probable. The extension of the classical definition by Neyman and Cramer to continuous probability, using the ratios of measures of sets, faces a similar difficulty. For with $f(x)$ monotonic, $f(x_2)-f(x_1)$ may be taken as the measure of the interval from x_1 to x_2 just as well as x_2-x_1. It takes a human judgment to select. In connection with probability as the ratio of measures of sets, Jeffreys might also have mentioned Lomnicki and Steinhaus, Fundamenta Mathematicae, 1923. As to probability as a limit, Jeffreys writes (p. 304): "The existence of the limit is taken as a postulate by Mises, whereas Venn hardly considered it as needing a postulate. Thus there is no saving of hypotheses in any case, and the necessary existence of the limit denies the possibility of complete randomness." In R. A. Fisher's "infinite population," there remains implicitly in the randomness the notion of "equally probable." In this respect, this infinite population resembles the Willard Gibbs "ensemble" (p. 300).

Jeffreys also criticizes certain applications of probability notions to particular problems. He considers (p. 309) that "Student" in his z-distribution found a direct probability instead of an inverse probability. He states (p. 323) "My main disagreement with Fisher con-

cerns the hypothetical infinite population which is a superfluous postulate since it does not avoid the need to estimate the chance in some other way, and the properties of chance have still to be assumed since there is no way of proving them. Another is that, as in the fiducial argument, an inadequate notation enables him, like 'Student,' to pass over a number of really difficult steps without stating what hypotheses are involved in them."

Jeffreys writes Bayes' theorem as: Posterior probability is proportional to the product of prior probability and likelihood. This likelihood corresponds to productive probability. He discusses in considerable detail the troublesome prior or "a priori" probability. For the range 0 to $+\infty$ he takes frequently the prior probability to be proportional to dv/v, although he calls attention (p. 100) to the fact that the integral of dv/v becomes infinite at both ends of the range.

Among applications, Jeffreys is perhaps most interested in significance tests, treated in Chapters V and VI, 107 pages. But after a preliminary discussion of direct probabilities, he takes up in Chapters II to IV: multiple sampling and the multinomial law, the Poisson law, normal law of error, Pearson laws, negative binomial law, characteristic function, t and z distributions, the method of least squares, rectangular distribution, correlation, maximum likelihood, approximation and successive approximation, expectations, effects of grouping, smoothing, rank correlation, grades, contingency, and artificial randomness.

In Appendix I are given certain tables for chi-square and t^2. And in Appendix II on the factorial function, a short table is given of the diagramma function, the derivative of $\log x!$; that is, of $\log \Gamma(x+1)$. There is an index jointly for topics and authors.

EDWARD L. DODD

NOTES

A conference in topology was held at the University of Michigan on June 24–July 6, 1940. The following lectures were given: *Triangulated manifolds and differentiable manifolds*, S. S. Cairns, Queens College; *Topological functions*, E. W. Chittenden, University of Iowa; *Uniformity in topological space*, L. W. Cohen, University of Kentucky; *Extension and classification of continuous mappings*, Samuel Eilenberg, University of Michigan; *The role of local separating points in certain problems of continua structure*, O. G. Harrold, University of Virginia; *Abstract complexes*, Solomon Lefschetz, Princeton University; *Extensions of homeomorphisms in the plane*, Saunders MacLane, Harvard University; *Periodic and nearly periodic transformations*, P. A. Smith, Barnard College; *Regular cycles in compact spaces*, N. E. Steenrod, University of Chicago; *Topology of differentiable manifolds*, Hassler Whitney, Harvard University; *Uniform local connectedness*, R. L. Wilder, University of Michigan; *Topological transformation groups and foundations of geometry*, Leo Zippin, Queens College. It is planned to have the University of Michigan Press issue, during the coming year, a volume containing these lectures, as well as abstracts of a number of papers which were delivered during the conference.

Johns Hopkins University has conferred the doctorate of laws on Dr. Vannevar Bush, President of the Carnegie Institution of Washington.

Professor Emeritus J. L. Coolidge of Harvard University has been awarded an honorary doctorate of laws by Harvard University.

Duke University has conferred the doctorate of laws on Dean L. P. Eisenhart of Princeton University.

Iowa State College has conferred the doctorate of laws on Professor Maria M. Roberts in recognition of fifty years of service to that college.

After twenty-nine years of service as Johnson Professor of Mathematics at the College of Wooster, Wooster, Ohio, Professor B. F. Yanney has retired with the title Johnson Professor of Mathematics Emeritus.

Dr. H. T. Muhly of Johns Hopkins University and Dr. C. E. Shannon of the Massachusetts Institute of Technology have been awarded National Research Fellowships in mathematics.

The William Lowell Putnam scholarship for study at Harvard University during 1940 has been awarded to Mr. A. M. Gleason of Yale University.

Professor Joseph Allen of the College of the City of New York will retire September 1, 1940.

Assistant Professor Max Astrachan of Antioch College, Yellow Springs, Ohió, has been made chairman of the department of mathematics.

Assistant Professor Wealthy Babcock of the University of Kansas has been promoted to an associate professorship.

Dr. R. P. Bailey of Lafayette College has been promoted to an assistant professorship.

Dr. N. H. Ball of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. F. S. Beale of Lehigh University has been promoted to an assistant professorship.

Associate Professor A. H. Black of Lebanon Valley College, Annville, Pennsylvania, has been promoted to a professorship.

Assistant Professor Florence L. Black of the University of Kansas has been promoted to an associate professorship.

Dr. Augusto Bobonis of the University of Puerto Rico has been promoted to an assistant professorship.

Mr. J. M. Boswell of the University of Kentucky has been appointed professor of mathematics and physics at Cumberland College, Williamsburg, Kentucky.

Dr. A. T. Brauer of the Institute for Advanced Study will lecture at New York University during 1940–1941.

Dr. F. L. Brooks of Kent State University has been promoted to an assistant professorship.

Associate Professor Jewell H. Bushey of Hunter College has been elected chairman of the department of mathematics.

Dr. F. A. Butter of the University of Southern California has been promoted to an assistant professorship.

Mr. R. B. Carpenter of the Metropolitan Life Insurance Company of New York has retired.

Dr. J. A. Clarkson of the University of Pennsylvania has been promoted to an assistant professorship.

Sister Rose Margaret Cook of Loretto Heights College, Loretto, Colorado, has been promoted to a professorship.

Assistant Professor H. V. Craig of the University of Texas has been promoted to an associate professorship.

Associate Professor M. M. Culver of the University of Pittsburgh has been promoted to a professorship.

Dr. E. H. Cutler of Lehigh University has been promoted to an assistant professorship.

Dr. C. H. Denbow of Ohio University has been promoted to an assistant professorship.

Assistant Professor H. T. Engstrom of Yale University has been promoted to an associate professorship.

Assistant Professor O. J. Farrell of Union College, Schenectady, New York, has been promoted to an associate professorship.

Assistant Professor W. W. Flexner of Cornell University has been promoted to an associate professorship.

Assistant Professor H. L. Garabedian of Northwestern University has been promoted to an associate professorship.

Assistant Professor O. H. Hamilton of Oklahoma Agricultural and Mechanical College has been promoted to an associate professorship.

Dr. E. E. Haskins of Northeastern University, Boston, Massachusetts, has been promoted to an assistant professorship.

Dr. A. E. Heins of Purdue University has been promoted to an assistant professorship.

Dr. J. D. Hill of Michigan State College has been promoted to an assistant professorship.

Dr. Banesh Hoffmann of Queens College has been promoted to an assistant professorship.

Dr. J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Assistant Professor B. W. Jones of Cornell University has been promoted to an associate professorship.

Assistant Professor Chosaburo Kato of Denison University, Granville, Ohio, has been promoted to an associate professorship.

Assistant Professor S. H. Kimball of the University of Maine has been promoted to an associate professorship.

Assistant Professor B. O. Koopman of Columbia University has been promoted to an associate professorship.

Dr. O. E. Lancaster of the University of Maryland has been promoted to an assistant professorship.

Assistant Professor V. S. Lawrence of Cornell University has been promoted to an associate professorship.

Associate Professor V. F. Lenzen of the University of California has been promoted to a professorship.

Assistant Professor Sophia H. Levy of the University of California has been promoted to an associate professorship.

Assistant Professor L. L. Lowenstein of Alfred University, Alfred, New York, has been promoted to an associate professorship and has been named chairman of the department of mathematics.

Dr. W. C. McDaniel of Southern Illinois State Normal University has been promoted to an assistant professorship.

Assistant Professor J. D. Mancill of the University of Alabama has been promoted to an associate professorship.

Dr. A. B. Mewborn of California Institute of Technology has been appointed to an assistant professorship at the University of Arizona.

Dr. T. W. Moore of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. Hermance Mullemeister of the University of Washington has been promoted to an assistant professorship.

Assistant Professor W. R. Murray of Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor Sara L. Nelson of Georgia State College for Women has been promoted to an associate professorship.

Dr. Emma J. Olson of Kent State University has been promoted to an assistant professorship.

Assistant Professor F. W. Perkins of Dartmouth College has been promoted to a professorship.

Assistant Professor G. W. Petrie of South Dakota State School of Mines has been promoted to an associate professorship.

Associate Professor D. E. Richmond of Williams College has been promoted to a professorship.

Assistant Professor J. B. Rosser of Cornell University has been promoted to an associate professorship.

Brother Louis de La Salle Seiler of St. Mary's College, Winona, Minnesota, is now dean of studies.

Associate Professor L. P. Siceloff of Columbia University has been promoted to a professorship.

Professor W. G. Simon of Western Reserve University has been made vice president.

Assistant Professor Ruth G. Simond of Hampton Institute, Hampton, Virginia, has been promoted to an associate professorship.

Professor Lao G. Simons of Hunter College has retired.

Mr. R. E. Smith of the University of North Carolina has been appointed to a professorship at Atlantic Christian College, Wilson, North Carolina.

Associate Professor A. H. Sprague of Amherst College has been promoted to a professorship.

Dr. A. E. Taylor of the University of California at Los Angeles has been promoted to an assistant professorship.

Associate Professor M. S. Vallarta of Massachusetts Institute of Technology has been promoted to a professorship.

Assistant Professor O. E. Walder of South Dakota State College has been promoted to an associate professorship.

Associate Professor Helen M. Walker of Teachers College, Columbia University, has been promoted to a professorship.

Dr. M. E. Wescott of Northwestern University has been promoted to an assistant professorship.

Assistant Professor Hassler Whitney of Harvard University has been promoted to an associate professorship.

Dr. C. R. Wylie of Ohio State University has been promoted to an assistant professorship.

Assistant Professor G. A. Yanosik of New York University has been promoted to an associate professorship.

During 1940–1941, Assistant Professor G. M. Ewing of the University of Missouri will be on leave at Princeton University.

During 1940–1941, Associate Professor A. R. Turquette of Florida Southern College, Lakeland, Florida, will be on leave of absence to study at Cornell University.

Dr. Olaf Helmer and Dr. H. E. Vaughan of the University of Illinois have been appointed associates in mathematics.

Dr. W. K. Morrill of Johns Hopkins University has been appointed an associate in mathematics.

The following appointments to instructorships are announced: University of Alabama: Dr. R. H. Bruck, Dr. R. E. Gaskell; Amarillo College, Amarillo, Texas: Dr. G. A. Whetstone; Antioch College, Yellow Springs, Ohio: Dr. Leonard Tornheim; University of Arizona: Mr. F. E. Hohn; Brown University: Mr. G. E. Forsythe; Carnegie Institute of Technology: Dr. R. F. Clippinger, Dr. A. D. Hestenes; University of Cincinnati: Dr. Gabriel Horvay; College of the City of New York: Dr. S. F. Barber; Harvard University: Dr. A. L. Whiteman; Hunter College: Dr. Annita Tuller; University of Illinois: Dr. P. H. Anderson, Dr. M. M. Day, Dr. K. L. Nielsen; Indiana University: Dr. F. J. Weyl; University of Indiana Extension, East Chicago, Indiana: Mr. J. F. Paydon; University of Maryland: Dr. H. E. Newell; University of Michigan: Dr. W. D. Duthie, Dr. Samuel Eilenberg, Dr. A. V. Martin; Michigan State College: Dr. J. F. Heyda; University of Missouri: Dr. G. E. Schweigert; Newcomb College, Tulane University: Miss Mary E. Ladue; Northwestern University: Dr. O. G. Harrold, Dr. W. T. Scott; University of Notre Dame: Dr. J. P. Nash; University of Pennsylvania: Mr. H. N. Laden; Princeton University: Dr. Brockway McMillan, Dr. A. D. Wallace; Purdue University: Dr. H. L. Langhaar, Dr. A. W. McGaughey; Reed College, Portland, Oregon: Dr. L. Louise Johnson, Mr. R. A. Rosenbaum; University of Saskatchewan: Dr. D. C. Murdoch; Stanford University: Dr. T. C. Doyle; Suffolk University, Boston, Massachusetts: Mr. F. X. Sutton; U. S. Naval Academy: Dr. W. E. Bleick; University of Washington: Dr. R. A. Beaumont, Dr. R. S. Phillips; University of Western Ontario: Dr. R. H. Cole; University of Wisconsin: Dr. A. O. Lindstrum.

It is reported that Professor S. Ortu Carboni of Genoa died November 5, 1939, at the age of seventy-nine.

Dr. R. S. Martin of the University of Illinois died June 2, 1940.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

394. Leon Alaoglu, Garrett Birkhoff, and B. J. Pettis: *General ergodic theorems.* II. Preliminary report.

As a further application of the methods of the paper *General erogdic theorems*, Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309, the following theorem is proved. The space E is a Banach space which is also a complete linear lattice; furthermore $\| |x| \| = \|x\|$, and the norm is a monotone non-decreasing function of positive elements. The abelian semi-group G consists of positive linear transformations on E to E; means of transformations of G are denoted by τ. Then, if $\limsup_\tau |\tau x| < +\infty$ for all x of E, the limit $\lim_\tau \tau x$ exists in order for all x for which it exists in norm, and the two are equal. These limits of means are defined on page 296 of the above paper. If E is reflexive, the hypothesis of lattice completeness can be dropped, and the limit in norm exists for all x. (Received June 19, 1940.)

395. M. M. Day: *Ordered sets and closures.*

A closure function on X is any function c defined on all subsets of X whose values are subsets of X; c is monotone if $Y \subset Y' \subset X$ implies $cY \subset cY'$. A set A is called effectively ordered if the order relation is transitive and if every element of A has a successor. A natural definition of convergence in a neighborhood space is given which reduces to the definition of the Moore-Smith limit if A is directed. Definitions of closure in terms of neighborhoods and of convergence of functions on effectively ordered sets are given and it is shown that the closures defined by either method are the monotone closures. Tukey's definition of order among directed sets is extended to effectively ordered sets and it is shown that much but not all of the theory that holds for directed sets carries over to effectively ordered sets. (Received June 8, 1940.)

396. M. M. Day: *Reflexive Banach spaces which cannot be made uniformly convex.*

The space $B=(B_1 \times B_2 \times \cdots)_p$ (S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 243) is shown to be reflexive if all B_i are. If B_i is taken to be the i-dimensional space with $\|b\|_{B_i} = \sup |b_i|$ or $= \sum |b_j|$, $j \leq i$, where $b=(b_1, \cdots, b_i)$, it is shown that B is not isomorphic to any uniformly convex space. By choosing subspaces isomorphic to these it can be shown, for example, that if $B_i = L^{q_i}$ or l^{q_i}, $1 < q_i < \infty$, and if there do not exist m and M with $1 < m \leq q_i \leq M < \infty$ for all i, then B is not isomorphic to a uniformly convex space. (Received June 10, 1940.)

397. H. L. Garabedian, H. S. Wall, and Einar Hille: *Hausdorff means and integral equations.*

If $\{a_n\}$, $\{b_n\}$ are two regular Hausdroff sequences, then $\{b_n\}$ is said to be a divisor of $\{a_n\}$ if there exists a regular Hausdorff sequence $\{c_n\}$ such that $a_n=b_nc_n$, $(n=0, 1, 2, \cdots)$. Let $f_a(x)=\sum a_n(-x)^n$, $f_b(x)=\sum b_n(-x)^n$. The authors show that $\{b_n\}$ is a divisor of $\{a_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that $f_a(x)=\int_0^1 f_b(ux)d\phi_c(u)$, $|x|<1$. By means of the Stieltjes inversion formula they show that this integral equation can be transformed into the equation of R. Schmidt, namely: $\phi_a(u)=\int_0^1\phi_b(u/v)d\phi_c(v)$ (which holds for all except possibly a finite or countable set of values of u), and obtain in this way a theorem of Hille and Tamarkin (Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 573–577). Application is made to prove the inclusion relation $(H_m, \alpha) \supset (H, \alpha+m-1)$, $(\alpha>0, m=2, 3, 4, \cdots)$, where (H_m, α) denotes summability defined by the mth row in the difference matrix in which the base sequence is the sequence $\{(n+1)^{-\alpha}\}$ defining summability (H, α). (Received June 6, 1940.)

398. Edward Kasner and J. J. De Cicco: *Equilong and conformal transformations of period two.*

The main problem of this paper is to find all equilong transformations of period two. In equilong geometry the set of all equilong transformations of period two may be classified into three distinct types: (T_1) equilong involutions, (T_2) K symmetries, and (T_3) D inversions. This is in contrast with the conformal theory where Kasner has proved that the set of all conformal transformations of period two consists of two distinct types: $(\mathfrak{C}_1)$ conformal involutions; and $(\mathfrak{C}_2)$ conformal symmetries. There are thus five distinct types of transformations of period two; three equilong and two conformal. The equilong transformations of period two may be reduced equilongly to the canonical forms: (T_1) symmetry in the positive y_1-axis $Z=-z$ where z represents the dual variable $x+jy$ with $j^2=0$ and (x, y) the hessian or equilong coordinates of a line, (T_2) symmetry in the origin accompanied by reversal of orientation $Z=\bar{z}$, and (T_3) symmetry in the x_1-axis accompanied by reversal of orientation $Z=-\bar{z}$. It is recalled that the conformal transformations of period two may be changed conformally to $(\mathfrak{C}_1)$ symmetry in the origin $Z_1=-z_1$ where z_1 represents the complex variable x_1+iy_1 with $i^2=-1$ and (x_1, y_1) the cartesian coordinates of a point, and $(\mathfrak{C}_2)$ symmetry in the x_1-axis, $Z_1=\bar{z}_1$. (Received June 5, 1940.)

399. Otto Szász: *On the jump of almost periodic functions and Fourier integrals.*

In a previous paper (Duke Mathematical Journal, vol. 4 (1938)) the author proved a theorem expressing the generalized jump of a periodic function $f(x)$ with period 2π as the limit of $(\pi/\log 2)\{\bar{\sigma}_{2n}(x)-\bar{\sigma}_n(x)\}$, as $n\to\infty$, where $\bar{\sigma}_n$ is the nth arithmetic mean of the series conjugate to the Fourier series of $f(x)$. Here analogous results are established for functions almost periodic in the sense of Besicovitch, and for Fourier integrals of functions in L^p, $1\leq p\leq 2$. (Received June 8, 1940.)

400. G. A. Whetstone and C. M. Cramlet: *On the invariance of the passivity conditions of systems of partial differential equations.*

The system of r linear differential equations in one dependent and n independent variables $X_iu\equiv a_i\partial u/\partial x_\rho\equiv 0, i=1, \cdots, r; \rho=1, \cdots, n$, is said to be complete if the Pois-

son operator (X_iX_j) applied to u yields a linear combination of the X_iu. Under arbitrary analytic transformations on the independent variables the property of completeness is invariant. A well known extension of this result is the theorem: The property that a system be involutory is invariant. The authors have extended these theorems to systems of equations with more than one dependent variable. In terms of a canonical system C which is in orthonomic form and assumed to be passive the general theorem which includes the foregoing classical theorems is: When the system C is transformed by an arbitrary analytic transformation on the independent variables and the new system reduced to an orthonomic canonical form $\bar{C}$, this new system will be passive. (Received June 17, 1940.)

401. Stefan Bergman: *Boundary values of functions satisfying partial differential equations of elliptic type.*

Let $U(z, \bar{z})$, $z=x+iy$, $\bar{z}=x-iy$, be regular in $|z|<2$ and let it satisfy there the equation $L(U)=U_{z\bar{z}}+\mathrm{Re}\,(AU_z)+CU=0$, where A is a complex function of $z, \bar{z}$ regular in $|z|\leqq 2$, C a real function regular in the same region, $U_z=\partial U/\partial x-i\partial U/\partial y$. Under these circumstances $\sum_{n=1}^{\infty}|a_n|^2n^{-1}<\infty$ is a sufficient condition that U has boundary values almost everywhere on $|z|=2$ for radial approach. Here $a_n=\int_0^{2\pi}[\tilde{U}(\rho, \phi)K_1^{(n)}(\rho, \phi)+\tilde{U}_\rho(\rho, \phi)K_2^{(n)}(\rho, \phi)]d\phi$ where $\tilde{U}(\rho, \phi)=U(\rho e^{i\phi}, \rho e^{-i\phi})$, $\tilde{U}_\rho(\rho, \phi)=\partial\tilde{U}/\partial\rho$, $0<\rho<2$ arbitrary. $K_k^{(n)}(\rho, \phi)$ are certain functions depending only on L and connected in a simple way with the solution $E=V(z, \bar{z}, \zeta, \bar{\zeta})\lg|z-\zeta|+W(z, \bar{z}, \zeta, \bar{\zeta})$ of $M(E)=0, z\neq\zeta$, M denoting the equation adjoint to L. (See Sommerfeld, Enzyklopädie der mathematischen Wissenschaften, II A, 7 c, p. 515.) (Received July 27, 1940.)

402. Stefan Bergman and D. C. Spencer: *On distortion in pseudo-conformal mapping.*

In the theory of conformal mapping, theorems concerning the change of measure of various euclidean quantities play an important role. For example, the area of certain sub-domains, as the theorems of Golusin and Bermant show, cannot be too small (G. M. Golusin, Comptes Rendus de l'Académie des Sciences de l'URSS, 1937, pp. 617–619; A. Bermant, Comptes Rendus de l'Académie des Sciences de l'URSS, 1938, pp. 137–140). This paper contains analogous results for pseudo-conformal mappings. The concept of "B-area" (Stefan Bergman, Rendiconti dell' Accademia Nazionale dei Lincei, vol. 19 (1934), pp. 474–478) is used, as well as a new representation for four-dimensional volume. The main result is an analogue in two complex variables of the theorem of Golusin. (Received August 1, 1940.)

403. R. P. Boas: *A general moment problem.*

Let $\{\phi_n(x)\}$ be a set of normalized functions belonging to L^2 on (0, 1). The object of this paper is to establish conditions on the ϕ_n under which any sequence $\{a_n\}$ of complex constants such that $\sum|a_n|^2<\infty$ can be represented in the form $a_n=\int_0^1 f(x)\phi_n(x)dx$, where $f(x)\in L^2$. It is found that a necessary and sufficient condition is that the quadratic form $\sum\sum x_m\bar{x}_n\int_0^1\phi_m(x)\bar{\phi}_n(x)dx$ should have a positive lower bound. When the ϕ_n are orthogonal, the condition is trivially satisfied, and the theorem reduces to the Riesz-Fischer theorem. Generalizations of the Hausdorff-Young-Riesz theorems are also obtained, and the theory is applied to some specific sets of functions. (Received July 29, 1940.)

404. R. P. Boas and D. V. Widder: *Functions with positive differences.*

Notation: $\Delta_\delta^0 f(x)=f(x)$, $\Delta_\delta^n f(x)=\Delta_\delta^{n-1}f(x+\delta)-\Delta_\delta^{n-1}f(x)$ $(n=1, 2, \cdots)$. A simplified proof is given of a theorem of T. Popoviciu (Mathematica, Cluj, vol. 8 (1934), pp. 1–85) that if $f(x)$ is continuous in (a, b) and if, for some $n\geqq 2$, $\Delta_\delta^n f(x)\geqq 0$ for all x and $\delta(>0)$ such that x and $x+n\delta$ are in (a, b), then $f(x)$ is of class C^{n-2} in (a, b) and has continuous right- and left-hand $(n-1)$th derivatives. It can be deduced from results of S. Bernstein that if $\Delta_\delta^n f(x)\geqq 0$ for all integers n in a sequence $\{n_k\}$ such that n_{k+1}/n_k is bounded, $f(x)$ is analytic in (a, b); an independent proof is given of this theorem. It is shown that if n_{k+1}/n_k is unbounded a function $f(x)$ can be constructed, of class C^∞, such that $f^{(n_k)}(x)>0$ $(k=1, 2, \cdots)$, while $f(x)$ is not analytic. (Received July 31, 1940.)

405. A. T. Brauer: *On a property of k consecutive integers.*

Pillai has just proved the following theorem (Proceedings of the Indian Academy of Sciences, Section A, vol. 11 (1940), pp. 6–12): In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of k integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k\geqq 17$. It is proved that this conjecture is true. (Received July 29, 1940.)

406. Herbert Busemann and Walther Mayer: *On the foundations of calculus of variations.*

The minimizing arcs of a two-dimensional variational problem with fixed end points need not be differentiable everywhere if the integrand $F(x, y; \dot{x}, \dot{y})$ is regular and continuous in all four variables. But if $F(x, y; \dot{x}, \dot{y})$ is regular and continuous and satisfies a Lipschitz condition with respect to x and y, all minimizing arcs are continuously differentiable. However, in general line elements will occur which are contained in no or in infinitely many minimizing arcs. (Received July 30, 1940.)

407. R. H. Cole: *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of collinear points.*

The expansion problem associated with the differential equation $u^{[n]}+P_1u^{[n-1]}+\cdots+P_nu=0$ whose solutions are conditioned relative to an arbitrarily given finite set of points on a fundamental interval by n linear boundary relations is a familiar one. It was studied by Wilder (Transactions of this Society, vol. 18 (1917)) who restricted his investigation to the equation in which the parameter λ appears only in the coefficient function P_n in the form $P_n=P_{n,0}(x)+\lambda^n$, and to boundary conditions which are free from λ. The present paper deals with the question of the convergence of the expansion when the functions P_k are taken as polynomials in λ with coefficients which are functions of x, and when the coefficients of the boundary relations are arbitrary polynomials in λ. The reduction of the given system to matrix form permits the adoption of the formal expansion of an arbitrary vector (that is, a set of n arbitrary functions) developed by Langer (Transactions of this Society, vol. 46 (1939)) for problems of this type. His convergence theorem, however, is presented under regularity restrictions which are not fulfilled by the problem at hand. A con-

vergence theorem is developed here which shows the uniform convergence of the simultaneous expansion of the n arbitrary functions under appropriate regularity conditions. (Received July 5, 1940.)

408. H. V. Craig: *On extensors.*

The first part of this paper is concerned with the transformation equations of certain quantities called "crossed extensors." The distinguishing feature of these equations is that the summations are not restricted to adjacent Greek and Latin indices. The second part contains certain applications of extensor methods to classical differential geometry. One of the results is such that it can be outlined briefly. Let ρ be the radius vector from a given origin to a point with coordinates x and let $x=x(t)$ be the equations of a parametrized arc. Obviously, ρ sup (M), the Mth derivative of ρ with respect to t, is a vector function of x sup $(\alpha)a$, $\alpha=0, \cdots, M$. The partial derivatives ρ sup (M) inf $(\alpha)a$ evidently transform cogrediently to an excovariant extensor. Consequently the scalar products of these vectors, denoted by g inf $\alpha a\beta b$, are the components of an extensor. Therefore the contraction of the metric tensor g sup bc with g inf αa Mc is a tensor-extensor which is represented by ρ sup b inf αa. The contraction of this quantity with V sup $a(\alpha)$, the derivatives of a contravariant vector V, over the reduced range $M-\beta$ to M, is the βth intrinsic derivative of V multiplied by a constant. From the extensor point of view, the tensor character of intrinsic differentiation is obvious without knowledge of the affine connection. (Received July 31, 1940.)

409. H. B. Curry: *A revision of the fundamental rules of combinatory logic.*

This paper is concerned with some general theorems concerning improvements in the formulation of the ultimate foundations of the theory of combinators. (Cf. the first part of the author's thesis, American Journal of Mathematics, vol. 52 (1930), pp. 509–525.) For the most part these improvements were suggested by the work of Rosser (cf. his thesis, Annals of Mathematics, (2), vol. 36 (1935), pp. 127–139) who formulated a weakened system of combinatory logic in which the rules had a simple character not possessed by the rules of the original system. This paper discusses ways of formulating the original system which are analogous to Rosser's formulation. A formulation is given based on the primitives of the original system, and also in terms of the Schönfinkel primitives, S and K. (Received July 27, 1940.)

410. H. B. Curry: *Consistency and completeness of the theory of combinators.*

Some theorems relating to the consistency and completeness of the theory of combinators were proved in the author's thesis (American Journal of Mathematics, vol. 52 (1930), pp. 509–536, 789–837). Later, analogous theorems were proved by Rosser (Annals of Mathematics, (2), vol. 36 (1935), pp. 127–150 and Duke Mathematical Journal, vol. 1 (1935), pp. 328–355); his theorems were, in some respects, more powerful, but they applied only to a weakened system. In this note it is shown that the strongest theorems can be extended to the full theory of combinators, with or without a normality condition similar to Rosser's. The methods involve some slight generalization of the methods used by Church and Rosser, including their theorem of conversion (Transactions of this Society, vol. 39 (1936), pp. 472–482). (Received July 30, 1940.)

411. John Dyer-Bennet: *A note on partitions of the set of positive integers.*

The partitions of the set of positive integers which are homomorphic with respect to addition are classified, and it is noted that they are also homomorphic with respect to multiplication. A necessary and sufficient condition that such a partition be homomorphic with respect to exponentiation is given. Elementary number-theoretic methods are used. (Received July 2, 1940.)

412. Benjamin Epstein: *A contribution to the theory of two complex variables in certain domains.*

In this paper the author investigates a.f. of 2 c.v. (analytic functions of two complex variables) in domains $\mathfrak{S}^4=\sum_{|\gamma|\leq 1}S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$, where $\mathfrak{S}^4$ is a domain of four-dimensional space which is the set sum of sectors $S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$ lying in the plane $z_2=\gamma$ and with boundary made up of two rays proceeding from the point $z_1=0$ and making angles $\theta_1(\gamma)$, $\theta_2(\gamma)$ respectively with the positive real axis. The following result is proved: Let $\mathfrak{g}^1$ be a point set of $|z_2|=1$ which has the property that the intersection of $\mathfrak{g}^1$ with every arc of $|z_2|=1$ of length Γ ($\Gamma\geq\Gamma_0$, Γ_0 depending only on $\mathfrak{S}^4$) contains a set of measure equal to or greater than $\Gamma/p(p\leq p_0<\infty)$; let $f(z_1, z_2)$ be a bounded a.f. of 2 c.v. regular in $\mathfrak{S}^4-E[z_1=0, |z_2|\leq 1]$; and let $\underline{\mathrm{Bd}}_{z_1\to 0}f(z_1, e^{i\phi_2})=0$, $e^{i\phi_2}\ \varepsilon\ \mathfrak{g}^1$, uniformly in ϕ_2. Then $\lim_{z_1\to 0}f(z_1, z_2)=0$, $|z_2|<1$. Other theorems of similar type can be proved. Investigations concerning a.f. of 2 c.v. in the domains $\mathfrak{S}^4$ have many applications, since it is often possible to reduce the study of a.f. of 2 c.v. in an arbitrary domain whose boundary hypersurface contains a segment of an analytic surface to the study of a.f. of 2 c.v. in a $\mathfrak{S}^4$-domain (cf. Bergmann, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 1–42). (Received July 30, 1940.)

413. G. C. Evans: *Surfaces of minimum capacity.*

The author considers the problem of finding for a given closed curve in space the surface cap which is of minimum electric capacity. For this it is assumed that the curve itself is of zero capacity, and that a sphere containing it can be mapped in a continuous one-one manner on itself in such a way that the curve is represented by a circle. A surface is found which satisfies the minimal condition by determining a certain two-valued harmonic function on a Riemannian manifold of two space sheets. The surface is a level surface of this function. It furnishes the minimum capacity among all surfaces which are made up of sufficiently smooth pieces and have the given curve as sole boundary. (Received July 29, 1940.)

414. H. T. Fleddermann: *The equality of measure functions.*

Besicovitch (Mathematische Annalen, vol. 98 (1927), p. 422) defines the density of a linearly measurable plane set at a point of it. In this definition he makes use of Carathéodory's measure. In this note, the definition is extended by replacing the Carathéodory measure by the Gross measure and by any general measure function. It is shown that if the density obtained by using the Gross measure is equal to 1, everywhere, then the density obtained with the Carathéodory measure is equal to 1, almost everywhere, and vice versa. It is further proved that under these conditions the Gross and Carathéodory measure of the set are equal. It follows from this and the fact that the Gross measure is a minimum measure that the density obtained by using

any general measure function lying between the Gross and Carathéodory functions is also equal to 1 and the measure of the set is the same as the Gross or Carathéodory measure. As the Carathéodory measure is used as an upper bound, a class of measure functions is obtained for which it is a maximum. (Received July 30, 1940.)

415. J. S. Frame: *The double cosets of a finite group.*

Let H be a subgroup of order h of a finite group G of order $g=nh$. When the permutation group G_H of degree n on the right cosets is written as a group of matrices and completely reduced, let the irreducible representations Γ_i of degree n_i, $i=1, 2, \cdots, r'$, appear with multiplicity μ_i as components of G_H, and let $c_i=1, -1$, or 0 according as Γ_i has a symmetric, an alternating, or no bilinear invariant. Let G contain r double cosets $K_t=HS_tH/d_t$, $t=1, 2, \cdots, r$, consisting respectively of $k_t=h/d_t$ right cosets HS_t which are permuted transitively among themselves when multiplied on the right by elements of H. Let N_H of these be self-inverse. Then after some preliminary theorems about the distribution of elements within the double cosets of G, two principal theorems are proved. The first, which generalizes a result proved by the author in a previous paper (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139), states that $N_H=\sum_{i=1}^{r'}c_i\mu_i$. The second, which proves and generalizes a theorem conjectured by the author in another paper (Duke Mathematical Journal, vol. 3 (1937), pp. 8–17), states that $n^{\nu-2}\prod_{t=1}^{r}k_t=(\prod_{i=1}^{r'}n_i^{\nu}\overline{P}_1P_1$, where $\nu=\mu_1^2$ and P_1 is an algebraic integer in the field of the characters of the components of G_H. (Received July 23, 1940.)

416. K. O. Friedrichs: *On hyperbolic differential operators.*

Uniqueness and existence of the solution of Cauchy's problem for linear hyperbolic differential equations can be based on the projection theorem in Hilbert space and the use of certain "mollifying" integral operators, which had been employed for elliptic differential operators in a previous paper of the author. "Hyperbolic" differential equations are here understood in a wider sense; they are written as a system of the first order and contain the classical wave equations as a special case. The differential operator is to be closed by extending it to a certain linear space of L^2-integrable functions. The values of the solution are prescribed on a spacelike surface; the conditions for these initial data are such that they persist in the process of propagation. (Received July 30, 1940.)

417. Orrin Frink: *Series expansions in linear vector space.*

If the set $P=\{p_n\}$ is closed and minimal in the real or complex Banach space B, it is shown that every element x of B has a uniquely determined series expansion (1) $\sum\xi_np_n$ in terms of P, called its *minimal expansion*, such that (I) the coefficient of p_n, in any sequence of linear combinations of P which converges to x, converges to ξ_n; (II) the coefficient ξ_n is a continuous additive function of x; (III) P is part of a biorthogonal system $\{p_n, f_n\}$ and conversely, if $\{p_n, f_n\}$ is biorthogonal, then $\{p_n\}$ is minimal; (IV) the partial sums of (1) are in one sense the best possible approximations to x. A condition is given which is sufficient that the series (1) converge absolutely to x. If P is normalized, it is shown that $\xi_n\to 0$. Semiregular summability methods are introduced, and it is shown that no such method sums (1) to the wrong limit, while at least one such method sums (1) to x if only a finite number of the coefficients ξ_n are zero. Applications are given to Schauder series and minimal polynomial series, including power series and Newton interpolation series, both real and complex. (Received July 27, 1940.)

418. Abe Gelbart: *On approximations of functions of two complex variables regular in multiply-connected four-dimensional domains.*

In functions of one complex variable it is shown that every analytic function $f(z)$ regular in a multiply-connected domain D^2 of n holes can be approximated uniformly by $(n+1)$ functions, the first n being rational with one singularity at a point in one of the holes, and the $(n+1)$st being a polynomial. With the aid of Bergman's integral formula for functions of two complex variables (Recueil Mathématique, vol. 43 (1936), pp. 851–862), it is shown that an analytic function of two complex variables, with n singularity surfaces in a four-dimensional region $\mathfrak{M}^4$, can be approximated uniformly in a multiply-connected closed region $\overline{\mathfrak{A}}^4$, interior to $\mathfrak{M}^4$ and excluding the n singularity surfaces, by $[n(n+1)+1]$ functions, the first $n(n+1)$ having just two singularity surfaces and the last being a polynomial. By imposing further conditions on the domain $\overline{\mathfrak{A}}^4$, it is shown that each of the $n(n+1)$ functions can be approximated uniformly in $\overline{\mathfrak{A}}^4$ by rational functions that have the same two singularity surfaces. (Received July 30, 1940.)

419. Louis Green: *Twisted cubics associated with a space curve.* II.

This paper is an extension of results of the author (American Journal of Mathematics, vol. 62 (1940), pp. 285–306) on local projective properties of a space curve Γ. It is found that Γ possesses at each of its points a two-parameter family of five-point cubics which are intimately related to the osculating quadrics of Γ. By means of these cubics, new configurations related to Γ are obtained and new characterizations are given to some known ones. A number of the cubics are found to have special properties of interest. (Received July 31, 1940.)

420. D. W. Hall and W. T. Puckett: *Arc-preserving transformations of a certain class of spaces.*

It is shown that the class of cyclic strongly arcwise connected continua (W. T. Puckett, this Bulletin, abstract 46-5-316) consists exactly of all cyclic locally connected continua A such that every arc-preserving transformation $T(A)=B$, where B is not an arc, is topological. (Received July 29, 1940.)

421. Marshall Hall: *Projective planes.*

An investigation of projective planes, especially non-Desarguesian types. Consideration of the generation of planes, particularly "free" planes. A study is made of projections, collineations, and universal configurations. (Received July 22, 1940.)

422. P. R. Halmos and R. A. Leibler: *Square roots of measure preserving transformations.*

Let T be a one-to-one, measure preserving (not necessarily continuous) transformation of the m-dimensional unit cube into itself. The problem of the present paper is to find conditions necessary and sufficient for the existence of another such transformation S, called a square root of T, for which $S^2=T$. A complete solution is given in case T has a pure point spectrum: in other words, in case there exists a sequence of numbers α_n and a complete orthonormal set of functions $f_n(x)$ in L_2 such that $f_n(Tx)=\alpha_n f_n(x)$. The following two theorems (indicating, respectively, the nature of the necessary and of the sufficient conditions) are typical. I. If T has a square root, then the multiplicity of -1 with respect to the point spectrum of T is even (is

0, 2, 4, $\cdots$, ∞). II. If T is metrically transitive and has a pure point spectrum, and if -1 does not belong to the spectrum of T, then T has a square root. (Received July 30, 1940.)

423. J. G. Herriot: *Nörlund summability of double Fourier series.* Preliminary report.

Grünwald (Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), pp. 343–350) has proved that the square partial sums s_{nn} of the double Fourier series of an integrable periodic function of two variables are summable $(C, 1)$ to the value of the function at any point of continuity. The author replaces the $(C, 1)$ method by a Nörlund method with nonnegative, nonincreasing coefficients and obtains necessary and sufficient conditions that the above result remains true. The necessity is proved by appealing to a theorem of Banach and Steinhaus on linear functionals. The sufficiency follows from Grünwald's result. In particular, it follows that the (C, α) method applied to square partial sums possesses the localization property if and only if $\alpha \geqq 1$. Again consider the Nörlund transforms t_{mn} of the partial sums s_{mn} obtained by a double Nörlund transformation $\{p_m q_n\}$, p_n and q_n being nonnegative and nonincreasing. Consider the limit of t_{mn} when $m, n \to \infty$ so that $m/n \leqq \lambda, n/m \leqq \lambda$ ($\lambda \geqq 1$, a fixed number). Necessary and sufficient conditions that this method of summability should possess the localization property are obtained. In particular, when $0 < \alpha, \beta \leqq 1$, (C, α, β) as above restricted possesses the localization property if and only if $\alpha = \beta = 1$. (Received July 25, 1940.)

424. Harold Hotelling: *Experimental determination of the maximum of an empirical function.*

In physical and economic experimentation to determine the maximum of an unknown function, for example, of a monopolist's profit as a function of price or of the magnetic permeability of an alloy as a function of its composition, the characteristic procedure is to perform experiments with chosen values of the argument x. each of which then yields an observation, subject to error, on the corresponding functional value $y = f(x)$. The values of x need, however, to be chosen on the basis of earlier experiments in order to make the determination efficient. The experimentation properly proceeds, therefore, in successive stages, with the values used at each stage determined with the help of the earlier work. The question, what distribution of x as a function of previous results should be used, is discussed in this paper on the basis of various hypotheses regarding the function, and further criteria. In particular, a conflict is shown to exist under some conditions between the criterion of minimum sampling variance and that calling for absence of bias. (Received August 1, 1940.)

425. Fritz John: *Discontinuous convex solutions of difference equations.*

This paper is a supplement to the author's paper (Acta Mathematica, vol. 71 (1939), pp. 175–189) which dealt only with continuous convex solutions. It includes as special cases results by A. E. Mayer (Acta Mathematica, vol. 70 (1938), pp. 57–62) and by H. P. Thielman (this Bulletin, vol. 46 (1940), p. 432). It is shown that a difference equation can have only a discontinuous convex solution if it has more than one continuous convex solution. A difference equation $\sum_k a_k f(x+k) = 0$ with constant a_k has a discontinuous convex solution when and only when the equation $\sum_k a_k x^k = 0$ has a positive real root. (Received July 25, 1940.)

426. Mark Kac: *On a question of Ramanujan.*

Let $d(n)$ denote the number of divisors of n. Ramanujan conjectured (*Collected Papers*, p. 333, Question 770) that the series $\sum_{k=1}^{\infty}(-1)^k[d(2k+1)]/(2k+1)$ is convergent. The present paper furnishes a short and simple proof of this statement. It turns out that the sum of the above series is $\pi^2/16$. (Received July 27, 1940.)

427. W. H. McEwen: *On the degree of convergence of a series associated with an integro-differential system.*

This paper deals with the expansion of a function $f(x)$ in terms of characteristic solutions of a given homogeneous integro-differential system involving a parameter. Denoting the partial sums of the series by $S_N(x)$ $(N=1, 2, \cdots)$, and letting m denote an arbitrary positive integer, hypotheses are obtained under which $f^{(k)}-S_N^{(k)}=0$ $(1/N^{m-k})$ uniformly on the interval of x in question, where $k=0, 1, \cdots, m-1$. (Received July 31, 1940.)

428. H. E. Newell: *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.* Preliminary report.

The matric differential equation $(d/dx)Y(x,\lambda)^{n\times n}=\{\lambda(\delta_{ij}r_j(x))^{n\times n}+(q_{ij}(x,\lambda))^{n\times n}\}\cdot Y(x,\lambda)^{n\times n}$ is discussed for the parameter λ large in absolute value, and for x in a finite simply connected region of the complex plane, within which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i\neq j)$, not zero, and on the boundary of which the coefficient functions may have poles, and the differences $r_i(x)-r_j(x)$, $(i\neq j)$, may have zeros. Under four distinct sets of conditions, three of which admit of meromorphic $q_{ij}(x,\lambda)$, regions of existence abutting a specified pole of an $r_j(x)$ or a specified zero of a difference $r_i(x)-r_j(x)$, $(i\neq j)$, are established for a solution of the form $P(x,\lambda)(\delta_{ij}\exp\{\lambda\int^x r_j(x)dx\})$, where $P(x,\lambda)$ reduces uniformly in x to the identity matrix when λ becomes infinite. Two of the sets of conditions are also sufficient that $P(x,\lambda)$ admit, for $|\lambda|$ large, of an asymptotic series expansion in λ^{-1}. The method employed is similar to that recently used by R. E. Langer in treating the case in which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i\neq j)$, bounded from zero. Finally, it is also shown that the results of both the present and Langer's papers apply to infinite as well as to finite x regions. (Received July 3, 1940.)

429. Rufus Oldenburger: *La teoria de los polinomios de orden superior.*

This paper will be published in Spanish in the scientific and engineering publication "Ingenieria" of the National University of Mexico, where it was given as a series of five lectures. In this paper, solutions are obtained for all of the open questions listed in the last paragraph of the author's article on polynomials (Annals of Mathematics, (2), vol. 41 (1940), p. 709), together with applications. (Received July 29, 1940.)

430. J. M. H. Olmsted: *Lebesgue theory on a Boolean algebra.* Preliminary report.

Any partially ordered linear space with an element 1 (cf. H. Freudenthal, Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 641) can

be extended to a commutative ring, the multiplicative unit of which is the element 1. This extension is the space of all functions, from real numbers to elements of the underlying Boolean algebra, decreasing monotonely from the one element of the algebra to the zero element. Such functions are analyzed for an arbitrary Boolean algebra with countable superior and inferior. Furthermore, multiplication of functions and the existence of a measure on the Boolean algebra give rise to a theory of the L^p spaces. The Nikodym theorem for absolutely continuous functions is again available. (Received July 8, 1940.)

431. F. W. Perkins: *Polygonal and polyhedral means of functions.* Preliminary report.

Let R be a finite region of euclidean space of three dimensions. Let $G_P(a_P)$ be a closed subregion of R bounded by a regular polyhedron with center at P and radius a_P, and let $G_P(\rho)$ be the closed subregion bounded by a concentric similar polyhedron of radius $\rho \leqq a_P$, similarly oriented. Let $U(P)$ be a function of P, real and continuous in R. The author considers (as functions of P and ρ) various means of u associated with $G_P(\rho)$, especially the arithmetic means of u on the vertices, on the radial lines drawn to the vertices, on the triangular areas bounded by these radial lines and the edges, on the faces, and over the volume. Various relations involving these means are established, some of which are of particular interest in the theory of harmonic and subharmonic functions. Similar methods can be used in the plane; here some parts of the theory are related to topics studied by Frazer (*On the moduli of regular functions*, Proceedings of the London Mathematical Society, vol. 36 (1934), pp. 532–546). (Received July 25, 1940.)

432. G. Pólya: *Sur l'existence de fonctions entières satisfaisant à certaines conditions linéaires.*

The present paper deals with an interpolation problem for entire functions. Given two sequences of complex numbers $\{a_n\}$ and $\{A_n\}$ and a sequence of nonnegative integers $\{\alpha_n\}$. Does there exist an entire function $F(z)$ such that $F^{(\alpha_n)}(a_n)=A_n$ for all n? The author shows the existence of a solution when the sequence $\{a_n\}$ is periodic of period p, $\alpha_{n+p}=\alpha_n+p$, $\alpha_n \leqq \alpha_{n+1}$, $\lim \left|A_n/\alpha_n!\right|^{1/\alpha_n}=0$, and there is no polynomial of degree less than p satisfying the first p homogeneous equations. (Received July 20, 1940.)

433. E. J. Purcell: *Space Cremona transformations of order* $m+n-1$.

Consider a curve C_n of order n having $n-1$ points on each of two skew lines d and d', and a curve C_m' of order m having $m-1$ points on each of d and d' (m, n any integers). A generic point P determines a ray through it intersecting C_n once in α and d once in β. P also determines a ray through it intersecting C_m' once in γ and d' once in δ. Lines $\alpha\delta$ and $\beta\gamma$ intersect in P', the correspondent of P in a Cremona transformation of order $m+n-1$. Special positions of the defining curves give rise to involutions. The paper will appear soon in this Bulletin. (Received July 29, 1940.)

434. J. F. Ritt: *On a type of algebraic differential manifold.*

For manifolds of systems of differential equations involving one unknown function, operations of addition, multiplication and differentiation are defined and in-

vestigated. The paper will appear in an early number of the Transactions of this Society. (Received June 27, 1940.)

435. L. B. Robinson: *Singular solutions of nonlinear functional equations.*

Little is known about nonlinear functional equations. The author has computed some singular solutions and gives a simple specimen. Given the function $u(x)=x^{\lambda}/c$. This function satisfies an equation of the type $u'(x)=P(x)\{u(x^m)\}^2$ since it satisfies (I) $u'(x)=\lambda c x^{-2m\lambda+\lambda-1}\{u(x^m)\}^2$ and the transformation $x=1/\xi$ sends this into (II) $w'(\xi)=\lambda c\xi^{+2m\lambda-\lambda-1}\{w(\xi^m)\}^2$. Equation (II) admits a solution holomorphic at $\xi=0$ and depending on one parameter. It also has the singular solution $w(\xi)=\xi^{-\lambda}/c$. (Received July 25, 1940.)

436. R. M. Robinson: *Stencils for solving* $x^2\equiv a \pmod m$.

A set of 272 stencils for solving $x^2\equiv a \pmod m$ by the method of exclusion has been constructed on Hollerith cards. This provides a practical method for solving such congruences; the solution is particularly simple if $m<3000$. (Received July 29, 1940.)

437. E. Rosenthall: *On the diophantine equation* $x^3+y^3=z^3+w^3$.

The letters A, B, C, denote arbitrary algebraic integers of the quadratic number field $K(\rho)$, ρ an imaginary cube root of unity; all other letters represent rational integers. The conjugate of any integer Q is denoted by $\bar{Q}$. The following theorem is proved: All rational integer solutions of the equation $x^3+y^3=z^3+w^3$ are given by $x=-p(a+b)/q$, $y=p(a-2b)/q$, $z=-p(c+d)/q$, $w=p(c-2d)/q$ where $a+b\rho=(\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C})\bar{A}\bar{C}$, $c+d\rho=(\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C})AB$ and it suffices to take q equal to the G.C.D. of e, f, g, where $\overline{AB}\cdot B\bar{B}-AC\cdot C\bar{C}=e+f\rho$ and $\overline{AC}\cdot\overline{AB}-AC\cdot AB=g+2g\rho$. The proof is obtained by operating on a multiplicative equation in $K(\rho)$. Complete rational integer solutions are also obtained for each of the following equations: $x^3+y^3+z^3-3xyz=u^3+v^3+w^3-3uvw$ and $x^3+y^3+z^3-3xyz=u^3$. (Received June 29, 1940.)

438. A. R. Schweitzer: *Concerning general abstract relational spaces.* II.

In continuation of a paper reported in this Bulletin (abstract 46-3-190) the author constructs postulates for relational spaces $S_{n+1}(G, H)$ $(n=1, 2, 3, \cdots)$ when G is any substitution group on $n+1$ elements and H is an arbitrarily selected subgroup of G, including G. Distinction is made between "absolute" and "relative" invariance of the generating relation K_H effective between two $(n+1)$-ads of elements; the latter type of invariance is based on correspondence between the elements of the two $(n+1)$-ads. The symmetrical and transitive relation K_H is relatively invariant under G and absolutely invariant under H but not absolutely invariant under any substitution which is not in H. When G is the symmetric group the author's previous theory is obtained. When G is the alternating group and H coincides with G, instances include a system of postulates equivalent to the author's descriptive system ${}^3R_2$ (American Journal of Mathematics, vol. 31 (1909), pp. 387–388). When G is arbitrary the space $S_{n+1}(G, G)$ on $n+1$ elements consists of a definite set of $(n+1)$-ads corresponding to the group G on these elements. Other examples are given. (Received July 30, 1940.)

439. D. J. Struik and Alexander Wundheiler: *Affine representation of systems of forces.*

The well known geometrical representations of systems of forces (Study, Ball, and others) have an arbitrary character. This paper attempts a systematic treatment by introduction of a set of sliding affine figures (vectors, rotors, vector-rotors, rotor-vectors, and so on) obtained by combination of the properties of direction, orientation and normalization. Under the metric group certain figures merge together assuming traditional forms. Coordinates are assigned to these figures on the basis of their relations to projective figures. Merging conditions under the metric group with reflections are studied in particular. (Received August 1, 1940.)

440. H. S. Wall: *Real power series bounded in the unit circle.*

The author investigates the class T of power series $f(x)=\sum c_i(-x)^i$ with real coefficients, radius of convergence greater than or equal to 1, and with the property that $M(f)=\text{l.u.b.}_{|x|<1}|f(x)| \leqq 1$. The principal result is the theorem that to each function $f(x)$ of T there corresponds a function $F(x)$ of T which is representable in the form $F(x)=\int_0^1 xd\phi(u)/(1+xu)$ as a Stieltjes integral with respect to a bounded monotone mass function $\phi(u)$, such that $f(x)=[f(0)-x+(1-x)F(z)]/[1-xf(0)+(1-x)F(z)]$, $z=4x/(1-x)^2$. As $f(x)$ ranges over T, let $F(x)$ range over the subclass T_0 of T. Then a function is in T_0 if and only if it has a continued fraction representation of the form $(1+\lambda_0)(1-\lambda_1)x/4+(1+\lambda_1)(1-\lambda_2)x/1+(1+\lambda_2)(1-\lambda_3)x/4+\cdots$ (a 4 appears in every other partial denominator), where $-1 \leqq \lambda_n \leqq +1$, $(n=0, 1, 2, \cdots)$. The sequence $\{\lambda_n\}$ is the sequence associated with $f(x)$ by the algorithm used by Schur. (Received July 5, 1940.)

441. J. H. C. Whitehead: *On incidence matrices, nuclei and homotopy types.*

W. Hurewicz (Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 125) has asked: "Are two closed manifolds homeomorphic if they are of the same homotopy type?" It appears, among other things, that two closed, polyhedral manifolds (using the word manifold in its narrowest sense), which are of the same homotopy type, need not be combinatorially equivalent, and need not even have the same nucleus (J. H. C. Whitehead, Proceedings of the London Mathematical Society, vol. 45 (1939)). In fact, two lens spaces of types (m, q) and (m, r) are of the same homotopy type provided they have the same intersection invariant, that is, provided $r \equiv \pm l^2 q \pmod{m}$ for some l. This corrects an error in my note *On certain invariants introduced by Reidemeister* (Quarterly Journal of Mathematics (Oxford), vol. 10 (1939)) and it has not even been proved that these are topological invariants. For example, combinatorially inequivalent lens spaces of types (7, 1) and (7, 2) are of the same homotopy type but have different nuclei. Whether or no they are homeomorphic remains an open question. Reidemeister's invariants are the same for polyhedra with the same nucleus. (Received July 5, 1940.)

442. P. M. Whitman: *Splittings of a lattice.*

A "splitting" of a lattice L is defined as a partition of its elements into two "splinters": a principal ideal and a dual of a principal ideal. Necessary and sufficient conditions for the existence of splittings are given. The properties of the "edge" along which the splitting is made are investigated, and conditions given in terms of the

nature of the edge and splinters that L be modular or distributive. Successive splittings form a lattice. The free lattice of n generators has a proper sublattice isomorphic to itself. (Received July 22, 1940.)

443. L. R. Wilcox: *A topology for semi-modular lattices.*

Let L be a semi-modular lattice (cf. Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505) which is atomistic and satisfies the chain conditions. Let the set P of points (or atoms) of L be a metric space. It is shown under two mild assumptions of continuity that the topology of P can be extended to L so that L is a Hausdorff space. Indeed, if δ is the metric of P, and d is the dimension function on L, then neighborhoods of any element $a \in L$ are defined in the following manner. Let $\epsilon<0$, and let $p_1, \cdots, p_k$ be a finite system of independent points whose sum is a; then $U(a; \epsilon; p_1, \cdots, p_k) \equiv [b; d(b)=d(a), \delta(p_i, b)<\epsilon]$, where $\delta(p_i, b)$ is the distance between the sets $[p_i]$ and $[p; p \leqq b]$. The purpose of the paper is to clarify and unify the foundations of projective, affine and euclidean differential geometry by lattice-theoretic methods. (Received August 1, 1940.)

444. Orla V. Wood: *On relations between solutions of the differential equation of the second order with four regular singular points.*

The differential equation of the second order with four regular singular points $(0, 1, a, \infty)$ has not been seriously studied, nor has a method been given of extending its integrals from one domain of existence to another. The author considers the difference equation arising from the recurrence relations satisfied by the coefficients in a series solution about the origin. The difference equation is homogeneous, of the second order, the coefficients being of degree zero, two, and four, respectively. The solution of this equation requires the determination of a form suitable to represent it at infinity. An asymptotic solution is found from consideration of a related difference equation which is normalized. Making use of this solution, a Barnes type contour integral is formed. The integral is shown to converge, and to be an analytic solution of the differential equation throughout the complex plane with a cut along the positive real axis. Application of Cauchy's residue theorem to the contour integral leads to an analytic continuation outside the unit circle of the function represented by the original series, and to a relation between solutions of the differential equation. (Received July 30, 1940.)

445. Alexander Wundheiler: *Some basic notions in meta-geometry.*

A figure is a set of points. A figure and a normal divisor of a group G determine a G-hyper-figure of the group G. Two hyper-figures are H-equivalent, H being a subgroup of G, if they can be associated in a way invariant under the group H. A class of H-equivalent hyper-figures determines an H-object under the group H. An H-object has definite coordinates in any coordinate system of the group H. Examples: figure—an attached vector; hyper-figure—a free vector; a free contravariant and a free covariant vector are equivalent under the metric group. (Received August 1, 1940.)

NEW PUBLICATIONS

CURTISS, D. R., and MOULTON, E. J. A Brief Course in Trigonometry. Boston, Heath, 1940. 8+118+17 pp.

DROSTE, H. W. Die Lösung angewandter Differentialgleichungen mittels Laplacescher Transformation. (Neuere Rechenverfahren der Technik, no. 1.) Berlin, Mittler, 1939. 35 pp.

HEFFTER, L. Grundlagen und analytischer Aufbau der projektiven, euklidischen und nichteuklidischen Geometrie. Leipzig and Berlin, Teubner, 1940. 8+199 pp.

HOGBEN, L. La Matematica nella Storia e nella Vita. 2 vols. Translated by F. Morra. Milan, Hoepli, 1939. 16+916 pp.

KASNER, E., and NEWMAN, J. Mathematics and the Imagination. New York, Simon and Schuster, 1940. 14+380 pp.

LANE, E. P. Metric Differential Geometry of Curves and Surfaces. Chicago, University Press, 1940. 8+216 pp.

LEROY, F. Cours d'Algèbre et d'Analyse. Vol. 2. Analyse. Paris, Vuibert, 1940. 352 pp.

LITTLE, N. See ROBBINS, C. K.

LITTLEWOOD, D. E. The Theory of Group Characters and Matrix Representations of Groups. Oxford, Clarendon Press, 1940. 8+292 pp.

LOVE, C. E. Elements of Analytic Geometry. 2d edition. New York, Macmillan, 1940. 12+159 pp.

MIDDLEMISS, R. S. Differential and Integral Calculus. New York, McGraw-Hill, 1940. 10+416 pp.

MORRA, F. See HOGBEN, L.

MOULTON, E. J. See CURTISS, D. R.

NEWMAN, J. See KASNER, E.

ROBBINS, C. K., and LITTLE, N. Tables for Calculus. New York, Macmillan, 1940. 22 pp.

ROBSON, A. An Introduction to Analytical Geometry. Vol. 1. Cambridge, University Press; New York, Macmillan, 1940. 14+409 pp.

SETH, B. R. Two Dimensional Potential Problems Connected with Rectilinear Boundaries. (Lucknow University Studies, no. 13.) Lucknow University, 1939. 124 pp.

STELSON, H. E. The Mathematics of Business. Boston, Houghton Mifflin, 1940. 14+464 pp.

Table of the First Ten Powers of the Integers from 1 to 1,000. New York, Work Projects Administration, 1939. 82 pp.

Tables of the Exponential Function e^x. New York, Work Projects Administration, 1939. 535 pp.